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CONTENTS

Freshman Mathematics as an Integral Part of Western Culture . . . . . Morris Kline 295

Coupon Collecting for Unequal Probabilities . . . . . Hermann von Schelling 306

Barbilian Geometry and the Poincaré Model . . P. J. Kelly 311

Mathematics in the Secondary Schools for the Exceptional Student . . . . . H. W. Brinkmann 319

Alternative Solution to the Ehrenfest Problem . . . F. G. Hess 323

Mathematical Notes . . . Victor Thébault, W. V. Parker, ... T. A. Newton, Free Jamison, W. R. Scott 328

Classroom Notes . . . . J. P. Ballantine, D. E. Richmond, ... M. S. Klamkin, T. D. Nagle, C. M. Fulton 337

Elementary Problems and Solutions 345

Advanced Problems and Solutions 350

Recent Publications 357

News and Notices 362

The Mathematical Association of America 366

February Meeting of the Louisiana-Mississippi Section . . . 366

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FRESHMAN MATHEMATICS AS AN INTEGRAL PART OF WESTERN CULTURE

MORRIS KLINE, New York University

1. Introduction. I would like to re-open the very hackneyed question of what to teach the liberal arts student. My excuse for doing this is that the answer which has been given for generations is equally hackneyed and, what is more to the point, a totally unsatisfactory one.

Let me state at the outset that by the liberal arts student I mean the one who does not intend to use mathematics in some profession or career, and is taking mathematics because the subject is supposed to contribute to a liberal education. Though much might be said about the proper freshman courses for students who intend to continue their mathematical training beyond this first year or who may take just one year of mathematics but will have to apply it in specialized physics, chemistry, and biology courses (*e.g.*, pre-medical students), the needs of these latter two groups of students will not be discussed here. The recommendations to be made in this article concerning the liberal arts students are based on the assumption that this group can be segregated from the others. This segregation can be effected even in small colleges without creating any serious administrative problems.

As defined above, the liberal arts students constitute only one group of freshmen. However, by far the greatest percentage of freshmen belongs to this group. Also, since the segregation that is presupposed is in accordance with interests rather than ability, the liberal arts group will contain some of the most worthwhile students. For these reasons, then, this group must be given the utmost consideration. Moreover, since many of these students will become leaders in our society they will determine the fate of mathematics in some areas. Hence there are selfish reasons too for being concerned about the knowledge and impressions of mathematics which these students will acquire.

2. Objections to the traditional courses. What have we been feeding the liberal arts students? The almost universal diet has been college algebra and trigonometry. I believe that these courses are a complete waste of time. What educational values are there really in exponents, radicals, logarithms, Horner's method, partial fractions, binomial theorem, the trigonometric identities, and the law of tangents, just to mention some of the conventional topics?

Let us, in fact, examine this material in terms of the values commonly claimed as warranting its inclusion in the curriculum. One hears most often that students learn to reason by studying this material. Leaving aside the question of transfer of training, a moot point at best, I maintain that one cannot teach reasoning with this material because the material does not permit it. Every one knows that it is impossible to give freshmen a satisfactory definition of an irrational number. Hence we not only beg many questions of logic in teaching operations with irrational numbers but we compound confusion by using irrational

numbers as exponents in the subject of logarithms. Of course similar difficulties are ignored in the use of irrational numbers as trigonometric ratios associated with angles. In college algebra we use many properties of continuous functions without proof, relying upon a graph to convince the student of the correctness of these properties. For example, we use the fact that if the polynomial $f(x)$ has opposite signs for $x=a$ and $x=b$ then $f(x)=0$ has a root for a value of x between a and b. And we use the fact that $f(x)=0$ has at least one root. Even where we present proofs very few students can follow some of them. For example, how many students follow the proof of the binomial theorem for positive integral exponents, let alone fractions? When we teach partial fractions we generally omit the proofs because so much time would be required to present them correctly. Likewise when we teach the addition formulas and other identities of trigonometry we do not present proofs for all values of the variables because these proofs would take an unconscionable amount of time. To sum up, in our presentations of college algebra and trigonometry we mix proofs, intuitive arguments, graphical evidence, and assertions accepted without proof. Nevertheless, we expect the student to learn all the material. How he can come through with an appreciation of and respect for proof as exemplified by this one year of mathematics is not at all clear to me. I would say that we have confused him.

A second argument for mathematics is its beauty. There is beauty in mathematics. However, the question of what is beautiful must be answered subjectively. I contend that if I were to select portions of mathematics that are beautiful, they would not come from college algebra and trigonometry. As a matter of fact we know that these subjects are taught not because they are beautiful but because they are the next step towards higher mathematics after the high school subjects. And even if portions, at least, of college algebra and trigonometry were beautiful, it would be most difficult to sell mathematical beauty to students who are psychologically indisposed. To sell this beauty is like trying to give away ten dollar gold pieces. People are too suspicious even to examine one.

A third argument presented for the usual material is that everybody makes some use of mathematics and hence must learn some of it. But who of us has used college algebra and trigonometry outside of his professional work? There is actually nothing in these subjects which is ever used by an educated layman. Of course there are significant applications of college algebra and trigonometry but not only do these have no bearing on daily life but they occur in the context of advanced physics and mathematics so that the student cannot see where these techniques are of use even to scientists and engineers.

The truth about college algebra and trigonometry is that these subjects comprise nothing but a series of dry, boring, unmotivated, disconnected, and, to the student, unimportant techniques. The subjects are taught as techniques and the students are expected to master and reproduce them in parrot-like fashion. The entire body of material is of no value to the non-specialist and no argument for it, such as mental discipline, will ever make such material palatable

or pregnant with significance. The only argument in favor of this material is that it is readily forgotten.

There is unfortunately too much evidence to support the above evaluation of college algebra and trigonometry as courses for the liberal arts students. Those of us who have taught college freshmen know how little they get from the material. We know that we have failed to reach them and that all we succeed in doing is to intensify their dislike of mathematics. We know that they leave our courses grateful only for the fact that the year is over and vowing never again to become involved with mathematics.

Indirectly and unintentionally many of us have supplied evidence that the above evaluation is correct. A recent advertisement for one of the new college algebra texts described it as a real innovation, "a humanized approach to mathematical concepts." Algebra was supposedly presented as a language, as a logical science, and as a branch of human endeavor. It was also described as a collection of techniques and as a collection of puzzles. I could readily appreciate these last two values but was curious as to how the others might be incorporated. I therefore made haste to examine this "novel" algebra text and found exactly two and one-half pages out of more than 400 devoted to these values of algebra. There was in fact exactly one paragraph on algebra as a language, one on algebra as a logical science, and one on algebra as a branch of human endeavor. The rest of the book was the same as any one of the 4097½ college algebras which have appeared since 1900. The point of this illustration is not that the author failed to present the values of algebra claimed in the advertisement, though, of course, he did fail. Rather, here is an author who sees the need to present these values of mathematics, who tries to do so through the medium of college algebra, and who succeeds only in showing that the material does not lend itself to the exposition of these values.

Other authors, presumably equally dissatisfied with the meaningless and poverty stricken material of college algebra and trigonometry, add historical notes. After a long chapter on the theory of equations they tell us that Descartes was born in 1596, that he died in 1650, and that he was a philosopher. Such feeble, ridiculous, and pediculous efforts to incorporate human values are more to be pitied than scorned.

By persisting in the teaching of college algebra and trigonometry to the liberal arts student, college teachers of mathematics have been committing a heinous crime against mathematics and humanity. We have been guilty of teaching brick laying instead of architecture, and color mixing instead of painting. Such teaching has discouraged latent interest in mathematics and has embittered many young people towards mathematics, and in many cases, towards all learning. More than that, it has cut the ground from underneath the subject so that it is gradually losing its place in the liberal arts curriculum. Apropos of the teaching of the conventional material is the remark Mephistopheles makes to Faust in Goethe's drama:

Is it life, I ask, is it even prudence,
To bore thyself and bore the students?

To this quotation I would add that even the devil speaks the truth sometimes.

3. Some recent innovations in freshman mathematics courses. In recent years serious attempts have been made to improve the standard offering and I should like to discuss some of these briefly. A reasonably common alternative to college algebra and trigonometry is the mathematical analysis or general mathematics course which selects material from college algebra, trigonometry, co-ordinate geometry, and the calculus. Though the attempt to give the student a somewhat broader picture of elementary mathematics can be commended, the contents of the texts that have been published cannot be, for in effect all that has been accomplished is to select techniques from four fields instead of two. Since the basic ideas underlying the fields from which the techniques are drawn are now more numerous and in the case of the calculus, more difficult, the student is left even more at sea as to what is accomplished by all these techniques. Rigor is hardly even envisaged as an objective in these courses.

Some teachers, who are quite willing to break sharply with traditional material, attempt to teach the basic and broad concepts of modern mathematics and hence concentrate on such topics as functionality, transformation and invariance, and groups, rings, and fields. These topics are indeed basic ones in mathematics and they do constitute some of the most beautiful themes. However they are far too sophisticated and abstract to mean anything to freshmen and their beauty makes no appeal to the uninitiated. Moreover, one could hardly say that these topics present a rounded view of the multifaceted role of mathematics in our civilization.

Other teachers, highly conscious of the absence of rigor in conventional college algebra and trigonometry courses, have attempted to supply it by teaching the foundations of the real and complex number systems. This material is over the heads of freshmen. Real numbers are so familiar to them that the problem of defining them and rigorously establishing their properties does not strike them as significant. Such courses must devote a great deal of time convincing students that they ought to learn what the entire mathematical world did not miss for thousands of years. Moreover, aside from the difficulties in a really rigorous approach to the irrational number, the material is not representative either of pure or applied mathematics.

To my mind some better efforts to meet the problem presented by the liberal arts student were made by Dresden of Swarthmore in his *Invitation to Mathematics* and—if I may so speak of a contribution in which I was personally involved—by Cooley, Gans, Kline, and Wahlert of New York University in their *Introduction to Mathematics*. These two books are by no means alike. Yet both indicated a willingness to break away from conventional patterns and to abandon techniques in favor of ideas. They differ in that Dresden's

book is devoted to a wide range of ideas of pure mathematics and to the presentation of rigorous proofs, whereas the second one is more descriptive, includes some applications of mathematics, and points out the significance of some of the creations in our culture. Because the style and level of Dresden's book were difficult for freshmen it did not meet wide approval. The *Introduction to Mathematics* did receive a very encouraging response as measured in terms of the number of adoptions and did stimulate other authors to write similar books. (One author liked the idea of this book so much that he took over the entire pattern and used many sections verbatim, acknowledging, of course, the inspiration of Euclid, Newton, and Einstein.) Both of these books were a step in the right direction. However, to my mind both still compromised too much with conventional patterns and failed to present adequately the significance of mathematics in our civilization.

4. The philosophy underlying the cultural approach to mathematics. Having criticized past efforts so strongly I know that the time has come to state what I think should be done. This I propose to do next. First, however, I should like to state the philosophy which guides what I propose. Actually the conviction that a particular course seems right comes first and one invents the philosophy afterwards to rationalize the conviction. Nevertheless, a wise philosophy argues eloquently for the course which abides by it.

My philosophy contains three principles. The first of these states that *knowledge is a whole* and that mathematics is a part of that whole. However the whole is not the sum of its parts. The present procedure is to teach mathematics as a subject unto itself and somehow expect the student who takes only one year of the subject to see its importance and significance for the general body of knowledge. This is like giving the student incomplete pieces of a very complicated jig-saw puzzle and expecting him to put the puzzle together. It follows from this principle that mathematics must be taught in the context of human knowledge and culture.

The principle I have stated is not at all unfamiliar or unrecognized by teachers. But either because they themselves are ignorant of the true place of mathematics in our civilization and culture or because they become too steeped in their own specialty they ignore the broad picture and expect the student to supply it.

My second principle is that *mathematics must contribute to the objectives of a liberal arts program*. Though it is difficult to state these objectives precisely one might say that they are to acclimate the student to the civilization and culture in which he lives, to increase his appreciation of what that culture contains, and to prepare him for life in his cultural environment. It follows from this principle as well as the first one that a mathematics course for liberal arts students must relate the subject to other branches of our culture.

Many teachers agree with this principle but conform to it in strange ways. They teach how to mix x pounds of coffee at 20¢ per pound with y pounds of

tea at 50¢ per pound to make 100 pounds of some unspeakable mixture. In very modern textbooks this application has been made more realistic. Coffee is one dollar per pound and tea, two dollars.

My third principle is a negative one but it seems to be necessary to include it. *Don't compromise with your objectives.* Choose material and a presentation which directly fulfill the purposes of your course. Don't let tradition dictate subject matter. Don't drag the important ideas in by the tail while actually stressing conventional techniques.

5. Sketch of the proposed course. I believe the above principles and objectives can be adhered to in a satisfactory freshman liberal arts course and I propose to sketch the outlines of such a course. Briefly described, the object of the course is to present mathematics as a major constituent of modern culture. The course treats significant ideas of mathematics and the influences of these creations on other branches of our culture.

The material is arranged in historical order. The purpose is not to present the history of mathematics. Rather, the historical order is roughly the logical order. Moreover, by following the historical order it is possible to present the circumstances which led to the creation of a mathematical idea and to show the influence it exerted. Modern civilization and culture are an accumulation, an amalgamation, and a fusion of contributions from many earlier civilizations and cultures. The historical order of events permits us to break apart the whole complex of modern ideas insofar as they relate to our subject and to examine the contributions one by one.

I shall illustrate how some topics are treated. The material begins with some facts about mathematics in Egypt and Babylonia. The primary object of this topic is to present the origins of mathematics and its uses in relatively primitive civilizations. At the same time one can emphasize the empirical nature of pre-Greek mathematics in order to prepare the student for the change which takes place in the classical Greek period. In this chapter familiar facts about number are reviewed and accepted on the same basis as the ancient peoples themselves used, namely, experience.

We then proceed to the classical Greek period (600 B.C.–300 B.C.) and emphasize the change in the nature of mathematical activity introduced by the Greeks. Abstraction, deductive proof from explicitly stated axioms, and the emphasis on geometry as opposed to algebra are the salient changes and these can be related to the nature of classical Greek thought and society. In this same period I review a few facts about Euclidean geometry and then show that the outstanding characteristics of this creation are precisely those of Greek philosophy and Greek art. In addition, the significance of Euclidean geometry for later ages as a model of rigorous reasoning is pointed out.

The next topic is the mathematics of the Alexandrian Greek period, which lasted from about 300 B.C. to about 600 A.D. The mathematics of this period

favored science and measurement. Trigonometry arose at this time as a step towards quantitative astronomy and so I teach the trigonometry of right triangles and the use of trigonometry in the measurement of the sizes and distances of the heavenly bodies. Other applications of the trigonometry of the right triangle such as navigation and the analysis of forces into components can be included.

The final topic in the Greek period centers about the Greek doctrines that nature is designed in accordance with mathematical laws and that man can fathom that design through the mastery of mathematics. The importance of Euclidean geometry and of the supreme Greek achievement in astronomy, Ptolemaic theory, as evidence for the rationality of nature, is discussed. This material also shows that mathematics made possible the first truly great astronomical theory and the first great scientific synthesis.

The transition from the Greek age to the Renaissance is made by giving a brief historical account of the state of affairs in medieval Europe, the role of mathematics in this period, and the importance for subsequent developments of the Catholic doctrine that nature is rationally designed by God and intelligible to man. It is also relevant to emphasize at this stage that a lack of interest in life in the physical world is detrimental to the growth of mathematics.

The historical movement known as the Renaissance is then sketched. The flow of Greek works to Europe gave the Europeans the opportunity to take up where the Greeks left off. The revived interest in the natural world suggested a host of problems. And the fusion of the Catholic doctrine that nature is the handiwork of God and therefore worthy of study with the Greek doctrine of the mathematical design of nature inspired the mathematicians and scientists to take up their chosen tasks with religious zeal.

Among developments of the Renaissance the first one treated is the creation and introduction of the heliocentric theory by Copernicus and Kepler. This particular creation involves no new mathematical ideas but it does give one a chance to show how, for purely mathematical reasons and despite weighty scientific and religious counter-arguments, the courses of astronomy, science, philosophy, and even religion were altered.

A second great development of the Renaissance originates with the work of the painters. These men sought to depict nature realistically and hence faced the problem of reproducing three-dimensional scenes on canvas. By applying Euclidean geometry to this problem the Renaissance artists developed the science of perspective. A few principles of this science are readily taught and the effect of this development on painting can be demonstrated by displaying and contrasting late medieval and Renaissance paintings. In the consideration of projection and section, the basic idea underlying their science of perspective, the painters raised questions which the mathematicians took over and answered by the creation now known as projective geometry. A few theorems of this

geometry can be taught partly to enlarge the student's knowledge and partly to show that a vast and vital branch of mathematics was inspired by the art of painting.

The usual material of co-ordinate geometry is the next topic. However it appears to me to be very much worth while to lead into this subject by showing how Descartes, lost in the intellectual storms of the seventeenth century, turned to mathematics to find a new approach to truth. Descartes isolated the principles of mathematical method and applied his "universal mathematics" to philosophy. He then applied the method to the study of curves and came up with a much needed method of handling all curves. The potentialities in co-ordinate geometry for the study of new curves and higher dimensional geometry can now be discussed.

Like many other courses this one emphasizes the notion of functionality. The introduction of this concept is motivated by first discussing Galileo's new view of the role of science, namely, to describe natural phenomena quantitatively. Such descriptions are best made in terms of relationships among variables; hence the importance of the function concept. However instead of using artificial and disconnected examples of functions one can use the laws of motion and gravitation. Moreover, to show that functional relations expressed as formulas can be extremely helpful to scientists one can derive new laws of motion from Newton's laws and the law of gravitation. Derivations employing only elementary algebra and yet yielding significant conclusions can be given. It is then possible to show how successful these laws and their implications were in describing motions on the Earth and in the heavens. The flight of projectiles and the motions of the planets around the sun are encompassed. So far-reaching was the range of these laws that they were called universal laws. The role of the calculus in deriving universal laws, at least insofar as the concept of instantaneous rate of change is concerned, can be included in this general topic.

The remarkable evidence provided by the universal laws of motion for a mathematically designed and lawful universe was acclaimed by all European intellectuals and inspired a rationalistic movement. Imbued with the conviction that Reason, personified by mathematics, could solve all of man's problems, the great minds of the eighteenth century undertook a sweeping reorganization of science, philosophy, religion, ethics, literature, and the social sciences. It is possible to explain how science became more dependent upon mathematics, that doctrines such as materialism and determinism were built upon mathematical and scientific foundations, that the source and nature of truths were re-examined, that new religious movements culminating in Deism resulted, and that poetry was deprecated in favor of prose. All these topics can be presented in concrete terms. In a few cases they can be assigned as outside reading.

Returning to mathematics proper one could treat next the trigonometric functions and apply them to the mathematical analysis of musical sounds. Using sound waves as a concrete analogy one could also include a somewhat loose and largely qualitative account of electromagnetic phenomena. Though the mathematical theory of these phenomena proved to be invaluable for the design of the

radio, the telephone, television, and other modern miracles of science, its value in organizing and interpreting a whole class of seemingly diverse natural phenomena is stressed. Further, since the physical nature of all electromagnetic phenomena and of radio waves in particular is completely unknown, it is possible to illustrate the meaning of the prevalent philosophical doctrine that in the last analysis our best scientific knowledge reduces to mathematical formulas. These are the nature of the physical world.

A major objective of the course is to introduce the fundamentals of statistics. However, it seems highly desirable to explain why statistical methods were sought and emphasized. Hence one might point out first that the rationalistic spirit of the Age of Reason had infused the social scientists with the desire to discover the universal laws of their domains. But the attempts to deduce these laws of man and society from *a priori* principles failed to produce realistic sciences. The social scientists then sought and created new mathematical techniques for the derivation of laws by working from statistical data. The way is now open to treat the elements of statistics and probability and to show how significant laws were derived with these techniques.

From the statistical techniques there resulted statistically likely conclusions which, nevertheless, seemed to apply as infallibly as did the mathematically deduced, necessary laws of the Newtonian era. A new philosophy, the statistical view of nature, arose to challenge the philosophy of determinism and has had grave implications for philosophy, religion, and science. These implications should be discussed.

We are now close to modern times and, in view of the rapid development of mathematics in this period, some selection of topics is forced upon us. Among many possibilities the notion of transfinite numbers and the value of this concept in providing a sound analysis of length, time, and motion seem to warrant inclusion. Even more important in this period is the remarkable creation of non-Euclidean geometry. Several modern texts now include this subject but why they avoid pointing out the implications of this creation is a mystery to me. In intellectual spheres no creation of modern times has been more revolutionary for, in effect, non-Euclidean geometry has taught us that there are no truths. Moreover, this creation caused a marked revision of our understanding of the nature of mathematics.

Without expecting to do any more than pander to popular interest one could show how non-Euclidean geometry is applied in the theory of relativity. The presentation should be, of course, purely qualitative and highly simplified.

The entire presentation closes with a discussion of the twentieth century's understanding of the nature of mathematics. The essential features, mathematics as a method of approach to knowledge and mathematics as an art, should be stressed.

It is perhaps needless to point out that many a good intention has gone awry in the execution. The handling of the above material must produce a coherent, rounded, sober presentation of elementary mathematics. There must be proper balance between the large ideas and concrete illustrations. There

must be precise assertions and proofs. However, there is no need to prove all the conclusions. Moreover, the proofs must be distinguished from intuitively grounded and loose arguments. Finally, the treatment must be on a level suitable for freshmen.

6. Criticisms of the proposed course and rebuttal. The above course does seem to be a radical departure from conventional ones and in discussions with colleagues I have encountered criticisms which I should like to consider for a moment. One such criticism is that the course would amount to just a lot of vague talk, an ill-defined mass of material. One answer to this criticism could be that no course is vaguer as to the nature and role of mathematics than college algebra and trigonometry. However, more to the point is it that each topic has definite mathematical content and establishes definite relationships of mathematics to our culture. The critic who believes that the kind of course I have outlined would be vague evidently doesn't believe that mathematics has any ideas or significance and hence that there would be little to say about it. A course which emphasizes concepts and their influences can be made substantial by insisting upon a good understanding of what is taught.

Again, I have been told that some of the topics are too sophisticated to be grasped by freshmen. This criticism is justified in part. However, each course must open up new fields and new concepts which will be pursued and more fully understood through work in subsequent courses. That is, each course must break fresh ground to some extent. In later courses some of these ideas will be reconsidered and the student will progress farther in these courses because he already has some notion of what is being discussed. The mathematics course I have sketched would break ground for science and philosophy courses in particular. More than that, many colleges are now experimenting with broad humanities courses in history, literature, physical science, and the social sciences. Each of these approaches our culture through its own avenues. The mathematics course described above would fit excellently into such a program and reinforce the other courses.

Some teachers would argue that the course I have outlined is mathematically thin. Where, for example, is the quadratic formula? Of course these critics are begging the question. What is mathematics? Evidently to these people it is a series of techniques. It does not include the meaning, purpose, and significance of the technical material. These critics merely reflect their own failure to see mathematics broadly. Better one idea well understood as to meaning, purpose, and significance than a thousand techniques however well mastered. A virtuoso on the violin is not a musician.

Moreover, concepts are more difficult to grasp than mechanical procedures which call for only parrot-like responses. Any child could learn that if $y=x^2$ then $dy/dx=2x$. However, how many calculus students can describe the concept represented by dy/dx and how many can state what is accomplished by the calculus?

Then there are "practical" objections. Some teachers point out that a student can't go on from a course such as I have described to more advanced mathematics. My answer is this. If a student who has indicated at the outset that he does not intend to pursue mathematics or use it in later life changes his mind at the end of the course, then he should be willing to take a technical course such as college algebra and trigonometry before going on to the higher courses. In any case the 99% who won't go on should not be sacrificed to the one per cent who might.

I know that some chairmen object to innovations in freshman mathematics because it is supposed to be hard for the student to transfer credit from one college to another. Hence they permit innovations as long as the first semester contains college algebra and the second, trigonometry. Other chairmen object to my course because it cannot be taught by the graduate students who, in some places called universities, teach the bulk of the freshmen. Of course these chairmen are putting the cart before the horse or are just rationalizing their unwillingness to consider new ideas.

There is a very real practical difficulty and that is to find college teachers of mathematics who will undertake to present the cultural aspects of mathematics. For various reasons which I propose to discuss at greater length at another time, mathematics teachers pay least attention to their most critical teaching problem, namely, the education of the liberal arts student. However, the task of obtaining the proper teachers is merely the task of recognizing and re-orienting ourselves to the problem posed by the liberal arts student. It is not the task of finding persons with extraordinary talents.

The best answer to all dubious objections is experimentation. This we are now doing at New York University. The material I have sketched was tried in one section of about 30 students during the academic year 1951–1952. Two sections were taught this material during the year 1952–1953. Two more sections of this course are presently being used for this experiment. The general cultural material on which the course draws has been gathered and published under the title of *Mathematics in Western Culture* (Oxford University Press, N. Y., 1953).

It is too early to assess fully the results of the experiment. One positive accomplishment can be noted. The students understand what is being tackled in each topic and participate fully in the class work. They feel that this material is their meat, so to speak, and partake of it. Since the course makes contact with painting, philosophy, literature, the social sciences, and other fields, even the most disinterested student is drawn into participation at some time. The interest displayed contrasts sharply with the reticence, resistance, and helplessness one encounters in teaching the traditional material.

7. The major objective of the proposed course. What should be the prime accomplishment of the proposed course? Essentially it should teach an appreciation of the role of mathematics in Western culture. Appreciation rather

than skill has long been recognized as an objective in literature, music, and art. It seems to me equally justifiable as an objective in mathematics, especially in view of the facts that interest in mathematics must be aroused and that the subject is more difficult to grasp.

If we are successful in teaching appreciation of mathematics we shall replace the present dislike and complete rejection of the subject on the part of students who have suffered through college algebra and trigonometry by respect and possibly liking for the subject. We may even succeed in inspiring them to maintain some contact with mathematics in later life and thereby secure for them one of the noblest of intellectual interests. We know that this interest will be well rewarded.

COUPON COLLECTING FOR UNEQUAL PROBABILITIES*

HERMANN VON SCHELLING, U. S. Submarine Base, New London, Connecticut

1. Introduction. Let us assume that a firm encloses in each package of a certain product one coupon which is chosen randomly from a stock of k different items. A customer who collects these coupons may gather some of them several times before he completes a whole set. The problem arises to determine the probability of obtaining all of the k coupons in n packages. An obvious generalization is what will be the probability that the lth different coupon is found in the nth package.

Since 1938 this question was treated by several authors [1], [2], [3] who always assumed that each coupon has the same probability $1/k$. The method was applied by M. G. Kendall and Babington Smith [4] for checking the randomness of their random sampling numbers.

In the general case the ith coupon has the probability p_i with $p_1+p_2+\cdots+p_k=1$. The solution of the general problem was published by the writer [5] in a German journal of a limited circulation as early as 1934. This paper remained practically unknown; the authors mentioned above did not refer to it. Besides, the formulas were given without proofs. Since the problem seems to be of a continuous interest, a short demonstration of the general case might be justified.

2. The general distribution. Let us assume k kinds of events which occur with the probabilities

$$p_1, p_2, \cdots, p_k; \qquad p_1 + p_2 + \cdots + p_k = 1. \tag{1}$$

* Opinions or conclusions contained in this paper are those of the author. They are not to be construed as necessarily reflecting the views or endorsement of the Navy Department.

Only one event can be observed at any one time. We are using the following notation:
At the n_1th observation the last of the missing events is noted,
at the n_2th observation the second last,
at the n_3th observation the third last,
. . . . . .
at the $n_{(k-1)}$th observation the $(k-1)$th last, *i.e.*, the second event is noted.

In order to illustrate this notation, let us throw a die twelve times with the following results:

$$3\ 5\ )\ 3\ 2\ )\ 5\ 4\ )\ 1\ )\ 2\ 2\ 3\ 1\ 6\ ).$$

The number 6 appears as the last one, on the 12th place, $n_1=12$;
the number 1 appears as the second last, on the 7th place, $n_2=7$;
the number 4 appears as the third last, on the 6th place, $n_3=6$;
the number 2 appears as the fourth last, on the 4th place, $n_4=4$;
the number 5 appears as the fifth last, on the 2nd place, $n_5=2$.

A symbol from the theory of symmetrical functions is to be used for getting compact formulas. The character

$$S_k f(p_1, p_2, \cdots, p_m)$$

means that the function f is to be formed for all $\binom{k}{m}$ combinations of the k indices indicated in the subscript of S and that all these terms are to be added. For instance

$$S_4 \frac{1}{p_1+p_2} = \frac{1}{p_1+p_2}+\frac{1}{p_2+p_3}+\frac{1}{p_3+p_4}+\frac{1}{p_4+p_1}+\frac{1}{p_1+p_3}+\frac{1}{p_2+p_4}.$$

This understood, the probability $w(n_m)$ that the mth last event is observed at the time n_m, reads

$$(2)\quad \left\{\begin{aligned} w(n_m)=S_k\Big\{&\binom{m-1}{m-1}(p_1+p_2+\cdots+p_m)(p_{m+1}+p_{m+2}+\cdots+p_k)^{n_m-1}\\ &-\binom{m}{m-1}(p_1+p_2+\cdots+p_{m+1})(p_{m+2}+p_{m+3}+\cdots+p_k)^{n_m-1}\\ &+\binom{m+1}{m-1}(p_1+p_2+\cdots+p_{m+2})(p_{m+3}+p_{m+4}+\cdots+p_k)^{n_m-1}\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &+(-1)^{k-m+1}(p_1+p_2+\cdots+p_{k-1})(p_k)^{n_m-1}\Big\}\\ &\text{with } k-m+1\leqq n_m<\infty. \end{aligned}\right.$$

Proof: The requested probability $w(n_m)$ is the sum of terms

$$p_{i_1}^{l_1} p_{i_2}^{l_2} \cdots p_{i_{k-m}}^{l_{k-m}} p_{i_{k-m+1}} \text{ with } l_1+l_2+\cdots+l_{k-m}=n_m-1.$$

All exponents l_j are positive. These terms occur in the sum

$$S_k(p_1+p_2+\cdots+p_m)(p_{m+1}+p_{m+2}+\cdots+p_k)^{n_m-1}.$$

However, this sum contains in addition terms of the same form for which one or more of the exponents l_j vanish. Let us consider the following expression in which the coefficients c_j are constant, *i.e.*, independent of the probabilities p_i:

$$\begin{aligned} S_k\{&c_1(p_1+p_2+\cdots+p_m)(p_{m+1}+p_{m+2}+\cdots+p_k)^{n_m-1} \\ &-c_2(p_1+p_2+\cdots+p_{m+1})(p_{m+2}+p_{m+3}+\cdots+p_k)^{n_m-1} \\ &+c_3(p_1+p_2+\cdots+p_{m+2})(p_{m+3}+p_{m+4}+\cdots+p_k)^{n_m-1} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+(-1)^{k-m+1}c_{k-m}(p_1+p_2+\cdots+p_{k-1})(p_k)^{n_m-1}\}. \end{aligned}$$

The coefficients c_j are to be determined so that all terms are cancelled which contain less than $(k-m+1)$ different probabilities p_i whereas all terms with exactly $(k-m+1)$ different probabilities p_i occur just once. These conditions yield the following system of $(k-m)$ linear equations. The first refers to the terms with only two probabilities, the second to the terms with three probabilities and so on.

$$(3)\quad \begin{cases} \binom{k-2}{m-1}c_1 - \binom{k-2}{m}c_2 + \cdots + (-1)^{k-m+1}\binom{k-2}{k-2}c_{k-m} = 0 \\ \binom{k-3}{m-1}c_1 - \binom{k-3}{m}c_2 + \cdots + (-1)^{k-m}\binom{k-3}{k-3}c_{k-m-1} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \binom{m}{m-1}c_1 - \binom{m}{m}c_2 = 0 \\ \binom{m-1}{m-1}c_1 = 1 \end{cases}$$

The solution of this triangular system is found to be

$$(4)\qquad c_j = \binom{m-2+j}{m-1}; \qquad j = 1, 2, \cdots, k-m.$$

In order to establish this result, we just have to solve the equations step by step, starting with the last one. Formula (4) completes the proof of the basic equation (2).

The probability $w(n_m)$ is a sum of geometrical progressions. Therefore it is

easy to find the cumulative distribution $W(n_m)$. We get

$$W(n_m)=S_k\left\{\binom{m-1}{m-1}(p_{m+1}+\cdots+p_k)^{k-m}-\binom{m}{m-1}(p_{m+2}+\cdots+p_k)^{k-m}\right.$$
$$\left.+\cdots+(-1)^{k-m+1}\binom{k-2}{m-1}(p_k)^{k-m}\right\}$$
$$-S_k\left\{\binom{m-1}{m-1}(p_{m+1}+\cdots+p_k)^{n_m}-\binom{m}{m-1}(p_{m+2}+\cdots+p_k)^{n_m}\right.$$
$$\left.+\cdots+(-1)^{k-m+1}\binom{k-2}{m-1}(p_k)^{n_m}\right\}.$$

If n_m goes to infinity, $W(n_m)$ converges to one and the second sum at the right side vanishes. Therefore the first constant sum must be unity. Indeed it can be proved that this sum equals $(p_1+p_2+\cdots+p_k)^{k-m}=1$. Therefore

$$W(n_m) = 1 - S_k\left\{\binom{m-1}{m-1}(p_{m+1}+\cdots+p_k)^{n_m}\right.$$
$$-\binom{m}{m-1}(p_{m+2}+\cdots+p_k)^{n_m}+\cdots \tag{5}$$
$$\left.+(-1)^{k-m+1}\binom{k-2}{m-1}(p_k)^{n_m}\right\}.$$

Let us note that $W(n_m)$ is simpler than $w(n_m)$. This is a decisive advantage of the "sequential approach." It is inherent to this method, as I have confirmed again and again with various distributions.

The arithmetic mean becomes

$$E(n_m) = S_k\left\{\binom{m-1}{m-1}\frac{1}{p_1+p_2+\cdots+p_m}\right.$$
$$-\binom{m}{m-1}\frac{1}{p_1+p_2+\cdots+p_{m+1}}+\cdots \tag{6}$$
$$\left.(-1)^{k-m}\binom{k-1}{m-1}\frac{1}{p_1+p_2+\cdots+p_k}\right\}.$$

This formula is not prohibitive anymore, if we confine ourselves to small values of k and m. The second moment may be obtained from

$$E\binom{n_m+1}{2} = S_k\left\{\binom{m-1}{m-1}\frac{1}{(p_1+p_2+\cdots+p_m)^2}\right.$$

$$-\binom{m}{m-1}\frac{1}{(p_1+p_2+\cdots+p_{m+1})^2}+\cdots \tag{7}$$

$$\left.+(-1)^{k-m}\binom{k-1}{m-1}\frac{1}{(p_1+p_2+\cdots+p_k)^2}\right\}.$$

Of course, the variance follows from

$$\operatorname{Var}(n_m) = 2\left[E\binom{n_m+1}{2}-\binom{E(n_m)+1}{2}\right]. \tag{8}$$

These formulas cover the general case completely.

3. The special case of equal probabilities. Since the case $p_1=p_2=\cdots=p_k=1/k$ was treated by several authors, it is sufficient to list the final formulas. From equation (2) follows

$$w(n_m) = \binom{k-1}{m-1}\frac{1}{k^{n_m-1}}\sum_{j=0}^{k-m}(-1)^j\binom{k-m}{j}(k-m-j)^{n_m-1}. \tag{9}$$

By using the so-called "differences of zero" we get

$$w(n_m) = \frac{(k-1)(k-2)\cdots m}{k^{n_m-1}}\left(\frac{\Delta^{k-m}0^{n_m-1}}{(k-m)!}\right). \tag{10}$$

Fisher and Yates [6] tabulated the function $(\Delta^n 0^s)/r!$. Therefore the probabilities $w(n_m)$ can be evaluated easily up to $n_m=26$, the range of Fisher and Yates' tables.

The cumulative distribution $W(n_m)$ becomes according to equation (5)

$$W(n_m) = 1-\binom{k-1}{m-1}\frac{1}{k^{n_m-1}}\sum_{j=0}^{k-m}(-1)^j\frac{1}{m+j}\binom{k-m}{j}(k-m-j)^{n_m}. \tag{11}$$

Simple formulas result for the arithmetic mean and the variance, namely the well known equations

$$E(n_m) = k\left(\frac{1}{k}+\frac{1}{k-1}+\cdots+\frac{1}{m}\right) \tag{12}$$

and

$$\operatorname{Var}(n_m) = k^2\left(\frac{1}{k^2}+\frac{1}{(k-1)^2}+\cdots+\frac{1}{m^2}\right)$$

$$-k\left(\frac{1}{k}+\frac{1}{k-1}+\cdots+\frac{1}{m}\right). \tag{13}$$

All these relationships agree with the results gained independently by the authors who were mentioned in the introduction.

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BARBILIAN GEOMETRY AND THE POINCARÉ MODEL

P. J. KELLY, University of California, Santa Barbara

1. Introduction. In a very brief paper in *Casopis Matematiky a Fysiky* (1934–1935), D. Barbilian defined a class of metric spaces (which we will call Barbilian spaces) and stated some of their properties. The paper is an outline of results and contains no proofs, but one of the stated properties is that a Barbilian space has unique geodesic connection of each pair of its points when and only when it coincides with the Poincaré model of hyperbolic geometry. From this point of view, general Barbilian spaces are not geometrically fruitful. However, it seems to the author that the Barbilian approach to the Poincaré model has certain advantages of simplicity and generality. The object of the present paper is to illustrate these in a short development.

2. Barbilian geometry. A metric space is a set of points such that to each pair of its points, a, b, there is assigned a real number $d(a, b)$ with the properties

(1) $d(a, b) \geqq 0$,
(2) $d(a, b) = 0$ if and only if $a = b$,
(3) $d(a, b) = d(b, a)$,
(4) $d(a, b) + d(b, c) \geqq d(a, c)$.

Now, in the euclidean plane, let K be a set of points forming the interior of a

region bounded by a simple, closed curve J (Fig. 1). There is, then, a euclidean distance between each pair, a, b, in K and we denote this distance by ab. The problem we wish to consider is that of assigning some new distance law, or metric, to the points of K.

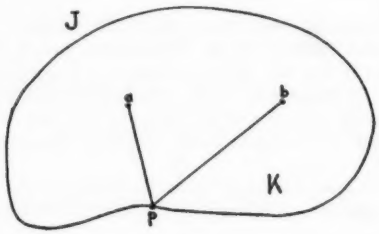

FIG. 1

Let a and b be particular points of K and let p be any point on J. As p varies on J, the distances pa and pb vary continuously and each assumes a maximum and a minimum value at some position of p. Any continuous function of pa and pb also has this property and so either the maximum or the minimum of the function associates a particular function value with the pair, a, b. What we seek is a function such that the value associated with a and b also satisfies the properties (1)–(4), that is, is a metric function.

To be specific, let the maximum be selected. The simplest functions of pa and pb we could consider are the basic arithmetic ones, $pa+pb$, $pa-pb$, $pa\cdot pb$, and pa/pb. That is, we inspect as possible distance functions

$$\max_{p\in J}(pa+pb), \qquad \max_{p\in J}(pa-pb),$$

$$\max_{p\in J}(pa\cdot pb), \qquad \max_{p\in J}(pa/pb).$$

All of these satisfy property (1). However, for $a=b$ the sum and product cases do not yield zero distance and can be discarded. The difference function is constantly zero for $a=b$ and so has zero as a maximum. It does not have the symmetry property (3), but this is easily corrected by taking

$$d(a,b) = \max_{p\in J}(pa-pb) + \max_{q\in J}(qb-qa).\dagger$$

It is not difficult to show that this function has the four basic properties and is a metric. However, under this metric, like the euclidean one, K is a bounded space, and there is a more interesting possibility.

† A slight generaliztion is obtained by using $kd(a,b)$ where k is a positive constant for all points a, b.

For $a=b$,

$$\max_{p\in J}(pa/pb) = \max_{p\in J}(pa/pa) = 1.$$

Thus, the quotient maximum is one when $a=b$. Because J is a simple closed curve, there are always points of J on each side of the perpendicular bisector of the segment from a to b. Thus for $a\neq b$, the quotient maximum is greater than one. These facts suggest the use of $\log [\max_{p\in J}(pa/pb)]$, which has properties (1) and (2), and can be modified, as before, to have property (3) by defining

$$d(a, b) = \log [\max_{p\in J}(pa/pb)] + \log [\max_{q\in J}(qb/qa)].$$

This is the metrization of K which Barbilian defined and which leads to an infinite metric space for any choice of J.

To investigate this metric, it will be convenient to define

$$r(a, b) = [\max_{p\in J} pa/pb]$$

$$r(b, a) = [\max_{q\in J}(qb/qa)]$$

so that

$$d(a, b) = \log [r(a, b)\cdot r(b, a)].$$

If a, b, and c are three points of K, we have,

(5) $$r(a, b) \geqq pa/pb \qquad \text{for all } p\in J$$

and

(6) $$r(b, c) \geqq pb/pc \qquad \text{for all } p\in J,$$

hence

(7) $$r(a, b)\cdot r(b, c) \geqq pa/pc \qquad \text{for all } p\in J$$

which implies

(8) $$r(a, b)\cdot r(b, c) \geqq r(a, c).$$

By a similar argument,

(9) $$r(b, a)\cdot r(c, b) \geqq r(c, a).$$

Combining (8) and (9), we have

(10) $$r(a, b)\cdot r(b, a)\cdot r(b, c)\cdot r(c, b) \geqq r(a, c)\cdot r(c, a),$$

and, upon taking the logarithm of both sides, this yields

(11) $$d(a, b) + d(b, c) \geqq d(a, c),$$

which establishes d as a metric.

In this last argument, it can be observed that the equality will hold in (11) if and only if the equality holds in both (8) and (9). Now let p^* be a maximizing point of J for $r(a, c)$. Then

$$r(a, c) = \frac{p^*a}{p^*c} = \frac{p^*a}{p^*b} \cdot \frac{p^*b}{p^*c} \tag{12}$$

shows that the equality in (8) holds if and only if p^* is also a maximizing point for $r(a, b)$ and $r(b, c)$. Similarly, the equality in (9) holds if and only if there is a common maximizing point q^* on J for $r(b, a)$, $r(c, b)$ and $r(c, a)$. That is, we have a result stated by Barbilian:

THEOREM: *Three points of K are collinear in the Barbilian sense if and only if their three distances can be defined by a common pair of points on J.*

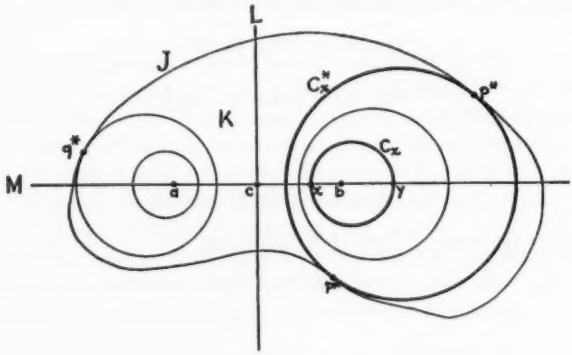

FIG. 2

For a given pair of points, a, b, in K, maximizing points on J, p^* and q^* for $r(a, b)$ and $r(b, a)$ respectively are determined very simply. Let c be the midpoint of the open, euclidean segment from a to b, namely $S^*(a, b)$ (Fig. 2). Take L to be the euclidean, perpendicular bisector of $S^*(a, b)$. If x is any point of $S^*(c, b)$, and y is the harmonic conjugate of x with respect to a, b, then the circle C_z, with the closed segment $S(x, y)$ as a diameter, is the locus of all points t such that $ta/tb = xa/xb = \alpha$. As x proceeds from c toward b, the circles C_z are decreasingly nested and α increases from 1 monotonically without bound. Because b is not on J there exists a circle C_{z_1} lying wholly in K. But all the circles C_z, for x on $S^*(c, b)$, fill (except for b) the half-plane defined by L and to which b belongs. It follows that as x proceeds from x_1 toward c there is a circle C_z^* of first contact with J. If p^* is any point on both C_z^* and J then $p^*a/p^*b = r(a, b)$. For if $C_{z_1} \subset C_z^* \subset C_{z_2}$, then C_{z_2} contains no points of J, while C_{z_1} corresponds to a smaller value of α than does C_z^*. By a similar argument, for x on $S^*(a, c)$, there

is a second circle, centered on M, separating a and b harmonically, and tangent to J at some point q^*. Then $q^*b/q^*a = r(b, a)$.

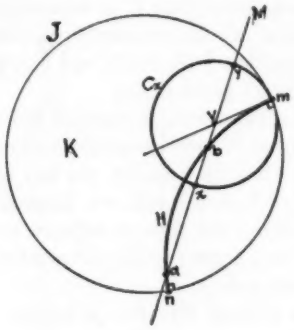

FIG. 3

Consider now the case where J is a circle and K is its interior (Fig. 3). If the line M through two distinct points a and b in K does not pass through the center of J, there is a unique circle H which passes through a and b and its orthogonal to J. Let m and n denote the intersection points of J and H, with b on the smaller arc $\widehat{am}$ of H. The radius of J to point m cuts M at a point v. Let C_v be the circle with center v and radius vm. Because C_v is tangent to J it is orthogonal to H and so separates a and b harmonically. Therefore m maximizes ta/tb for t on J. Similarly n maximizes tb/ta for t on J. Thus,

$$d(a, b) = \log [(ma/mb) \cdot (nb/na)],$$

which is also the Poincaré definition for distance. The points m and n are, by the same argument, the maximizing points for all pairs a and b on H, hence H is a geodesic. Because of the theorem established previously, H is the unique geodesic connecting a and b. When a and b are collinear with the center of J, the diameter of J containing a and b is simply a limiting position for H and the same results hold.

In this development, the cross ratio form $(pa/pb)(qb/qa)$ for distance appears in a natural way. Since this form is invariant under inversion, it is easily shown that inverting with respect to a point outside J sends K to an isometric space K'. The whole exploratory approach suggests that there may be other alternatives and it is natural to consider the possibility of replacing the maximizing points on J by the intersection points of J with the line through a and b. This leads, of course, to a Hilbert geometry, if J is convex, and gives the Klein model of hyperbolic geometry when J is an ellipse.†

† The comparison of Barbilian, Hilbert, and Poincaré spaces was pointed out by L. M. Blumenthal, *Distance Geometries*, 1938, pp. 27–28.

3. A generalization. Let a and b be two distinct points of K with a to the left of L, the perpendicular bisector of the segment from a to b. The ratio pa/pb, $p \neq b$, is less than, equal to, or greater than one according as p is to the left, on, or to the right of L. The condition that K be interior to a closed curve J simply insures that $r(a, b)$ and $r(b, a)$ are determined by points to the right and left of L respectively, and hence that $\log [r(a, b)r(b, a)]$ is a positive number. But this can be accomplished in a more general way.

Suppose J is any closed set which is disjoint from a set K to be metrized, and take a, b and L as before. To the previous definitions of $r(a, b)$ and $r(b, a)$, add the condition that if J has no point to the left of L then $r(b, a) = 1$ and if J has no point to the right of L then $r(a, b) = 1$. Since $\lim_{ta \to \infty} ta/tb = \lim_{ta \to \infty} tb/ta = 1$, this amounts to adjoining the line at infinity to J. With one exceptional case, namely when J is L or a subset of L, the product $r(a, b) \cdot r(b, a)$ is always greater than one for distinct points.

The new metric can be defined directly by taking

$$r(a, b) = \max \{1, \max_{p \in J} pa/pb\}$$

$$r(b, a) = \max \{1, \max_{q \in J} qb/qa\}$$

$$d'(a, b) = \log [r(a, b) \cdot r(b, a)].$$

The validity of $d'(a, b)$ as a metric can be established, save for the exceptional case, by much the same arguments as before, and when J, K define an ordinary Barbilian space then d and d' are the same. One advantage of the more general definition concerns the inversion of an ordinary Barbilian space with respect to a point of J. The image sets, J' and K', define an isometric space with respect to d' but not with respect to d. For example, when J is a circle with interior K, then inversion with respect to a point of J yields, under d', the familiar half-plane representation of hyperbolic space.

4. An example. It is easy to exhibit some particular geodesics in an ordinary Barbilian space. For if C is any circle, containing no point exterior to J, and touching J at m and n, then the circular arc in K orthogonal to C at m and n is a Barbilian geodesic.

To students who have seen the difference in character between euclidean and hyperbolic parallels, it is usually interesting to examine a space in which both types occur, and to see that this is achieved at the expense of unique geodesic connection. As an example, let K be the set of all points interior to either of two, equal-sized, orthogonal circles C_1 and C_2, which intersect at points m and n (Fig. 4). Then J consists of the larger arcs $\overset{\frown}{mn}$ of C_1 and C_2. The points interior to both circles form the "lens" and the smaller arcs $\overset{\frown}{mn}$ will be called the "lens arcs." If o_1 and o_2 are the centers of C_1 and C_2 respectively, then every point of $S(o_1, o_2)$ is the center of a circle C which passes through m and n and which possesses no point exterior to J. The circular arc which is orthogonal to C

at m and n, and which lies inside the lens, is therefore a Barbilian line. In particular, the segment $S^*(m, n)$ is also a straight line in the Barbilian sense.

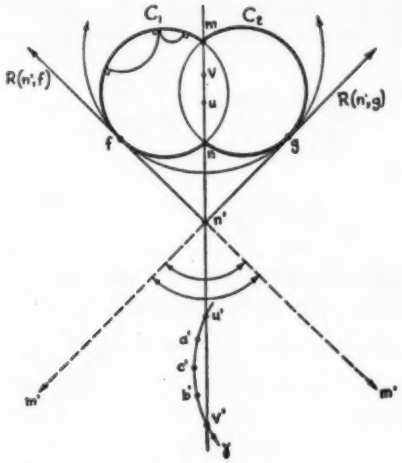

FIG. 4

Consider now an inversion with respect to a circle which is centered at m and which is tangent to C_1 and C_2 at f and g respectively. The transformation sends J into $R(n', g)$ (the ray through g which emanates from n') and into the perpendicular ray $R(n', f)$. These rays separate the plane into two regions and K' is the non-convex portion. The former family of circular arcs through m and n goes into the pencil of rays through n' which lies within the angle which is vertical to $\angle fn'g$. Let u and v be any two points of $S^*(m, n)$ and take γ to be a curve through u' and v'. If at all points of $\widehat{u'v'}$ on γ the curvature of γ is sufficiently small, it is apparent that for any pair of points a' and b' on $\widehat{u'v'}$ the perpendicular bisector of $S(a', b')$ will not intersect J'. If a' is nearer to n' than is b', then $d(a', b') = \log (n'b'/n'a')$. For any point c' of γ between a' and b', we then have $d(a', b') = d(a', c') + d(c', b')$, since $n'b'/n'a' = (n'c'/n'a')(n'b'/n'c')$. Thus the arc $\widehat{u'v'}$ of γ is a straight line segment in the Barbilian sense. There are therefore infinitely many straight line connections of u' and v' (including $S(u', v')$). Since the mapping is an isometry, there are therefore infinitely many straight lines connections of u and v.

The circular arcs which are orthogonal to both lens arcs, and which lie in the lens, invert into the arcs of concentric circles which are centered at n'. These are easily shown to be euclidean parallels. However, they are only Barbilian segments. The lens arcs themselves are examples of complete lines which are parallel in the euclidean sense. To see this, let o be a fixed point of the lens

arc which lies on C_1 and let x be a variable point on the lens arc which lies on C_2 (Fig. 5). Take p to be the intersection of J and the circle H_x which passes through o and x and which is orthogonal to C_2. Let q be the intersection of J with the circle M_x which passes through o and x and which is orthogonal to C_1. Then the Barbilian distance from o to x is given by

$$d(o, x) = \log\left(\frac{px}{po}\,\frac{qo}{qx}\right).$$

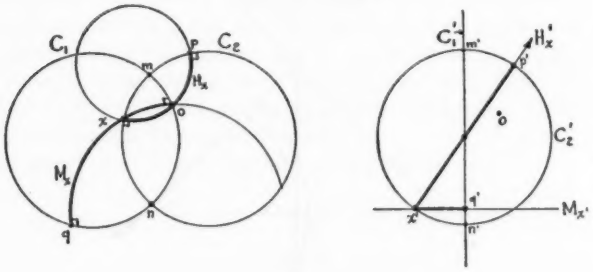

FIG. 5

If primes indicate the image points under an inversion centered at o, then from $px/po = x'p'/x'o$ and $qo/qx = x'o/x'q'$ we also have

$$d(o, x) = \log\,(x'p'/x'q').$$

Under the inversion, C_2 is transformed to a circle C_2'. Since C_1, H_x, and M_x pass through o, they invert into straight lines C_1', H_x', and M_x'. Because inversion preserves angles, it follows that C_1' is orthogonal to C_2' and therefore it passes through the center of C_2'. For the same reason, H_x' passes through the center of C_2', while M_x' is perpendicular to C_1'. The lines H_x' and M_x' intersect at x' on an arc $\widehat{m'n'}$ of C_2'. The point p' is the other intersection of H_x' and C_2', while C_1' and M_x' intersect at q'. As x varies, $x'p'$ is always the diameter length of C_2', $x'q'$ is always half the length of a chord perpendicular to the fixed diameter C_1'. The ratio $x'p'/x'q'$ is therefore a minimum when $x'q'$ is a maximum. This occurs when $x'q'$ is the radius length of C_2', that is when $d(o, x) = \log 2$. Since this result is independent of the position of o on the lens arc, the two lens arcs are equidistant straight lines in the Barbilian geometry at the constant distance log 2, and are therefore parallels in the euclidean sense. On the other hand, any two circles which are orthogonal to J at a common point of J (see Fig. 4) are Barbilian parallels which behave as do hyperbolic parallels.

In the construction just given, it is clear that the minimum for $d(o, x)$ occurs when H_x' and M_x' coincide, that is when H_x' is perpendicular to both C_1' and C_2'. In the original figure, then, the circle through o which is orthogonal to

both C_1 and C_2 carries the Barbilian perpendicular to both lens arcs. This, with the previous remarks, shows that the two conjugate systems of circles, defined by the base points m and n, form a gridwork of lines within the lens. It is also clear that inversion with respect to any circle which is orthogonal to both C_1 and C_2 is a motion of the space.

MATHEMATICS IN THE SECONDARY SCHOOLS FOR THE EXCEPTIONAL STUDENT‡

H. W. BRINKMANN, Swarthmore College

1. Introduction. The work to be described in this paper was done under the auspices of the School and College Study of Admission with Advanced Standing. This Study, which was initiated by President Gordon Chalmers of Kenyon College, was undertaken by twelve colleges* and twenty-seven secondary schools. Financial support for the Study was given by The Fund for the Advancement of Education, and Dr. William H. Cornog, President of the Central High School, Philadelphia, Pennsylvania, served as its Executive Director. The fundamental principle behind the study is that certain bright students are wasting time in the schools as set up at present, and that they could profitably spend some time in anticipating the work they would normally do in their first year of college. It was proposed that the twelve colleges should reach agreement on the content of the central subjects of the college freshman year, with a view to granting advanced credit in these subjects to those students who qualified in them. It was inevitable that mathematics be one of these subjects.† A committee was appointed to deal with each of these subjects and to define the subject matter concerned. This was done in consultation with representatives of the various institutions, and many lengthy meetings were held by each committee throughout the year to discuss the problems raised by the study. The committee for mathematics consisted of the following: Julius Hlavaty (Bronx High School of Science, New York City); Elsie Parker Johnson (Oak Park and River Forest High School, Oak Park, Ill.); Charles Mergendahl (Newton High School, Newtonville, Mass.); George B. Thomas (Massachusetts Institute of Technology);

‡ Presented to the Mathematical Association of America on December 31, 1953.

* Bowdoin College, Brown University, Carleton College, Haverford College, Kenyon College, Massachusetts Institute of Technology, Middlebury College, Oberlin College, Swarthmore College, Wabash College, Wesleyan University, Williams College.

† The other subjects were: English Composition, Literature, Physics, Chemistry, Biology, Latin, French, German, Spanish, and History.

Elbridge P. Vance (Oberlin College); Volney H. Wells (Williams College), with the writer (Swarthmore College) as chairman. The reports of all the committees are available at the office of Dr. Cornog, the Executive Director of the Study.

It should be mentioned here that the twelve colleges have all adopted the proposed plan to grant advanced credit for the work as outlined by the committees. Similar committees are now at work constructing examinations, which will be used to test the candidates for such credit. Furthermore, seven specially selected schools* are testing out these ideas by preparing candidates in several of the subjects mentioned. Many of the other cooperating schools are also giving courses of the sort proposed by our committees.

The problem before our committee was to define the work in mathematics which would be acceptable by our twelve institutions for college credit in place of a first-year college course. On the basis of consultation with the departments of mathematics of the twelve institutions, our committee decided that such a first-year course should consist essentially of a substantial course in calculus with applications, along with the analytic geometry that is needed for such a course. Our committee did not want to recommend that advanced credit be given for courses in college algebra, solid geometry, and the like, even though such courses are occasionally given in the freshman year at some colleges. When we attempted to integrate our freshman course into the High School program we found that it was necessary to give consideration to the whole program in mathematics in the secondary schools. We accordingly worked out a detailed program in mathematics for the last three years of secondary school, which culminates in a course that will be acceptable for advanced standing in college. When we got through with this we found that such a program is preferable in many ways to the standard program in mathematics as it is now pursued in most secondary schools, although we started out by devising it for the exceptional student.

2. The problem. It has been realized for many years that the mathematics curriculum in the secondary schools was in for a drastic revision. For example, the section on mathematics in *General education in school and college* (reprinted in this MONTHLY, vol. 60, 1953, pages 380–383) deals with this topic, and certain general proposals for revising the curriculum are made there. It will be seen that the program planned by our committee agrees in many ways with these proposals; in addition, we have given detailed suggestions as to how such a program can be carried out.

The main ideas behind our plan are: to break down the standard compartmentalized program for these years, to introduce certain new subject matter, and at the same time to suggest the elimination of certain traditional items. As

* Bronx High School of Science (New York City); Central High School (Philadelphia, Pa.); Evanston Township High School (Evanston, Ill.); Germantown Friends School (Philadelphia, Pa.); Horace Mann School (New York City); Newton High School (Newtonville, Mass.); St. Louis Country Day School (St. Louis, Mo.).

a background for this proposed program we assumed a course of instruction carrying the student through the ninth grade; he would thus be acquainted with the number system, the vocabulary and ideas of elementary, intuitive geometry, the beginnings of the use of algebraic symbolism, and the ideas of graphical representation. Our program for the three years of senior High School would then be the following:

(1) A course in 10th year mathematics stressing deductive thinking, but dealing with various types of mathematical subject matter.

(2) A year's course consisting of a continuation of algebra, of analytic geometry, and of trigonometry.

(3) A year's course made up of certain advanced topics in algebra and analytic geometry, with a substantial introduction to calculus and its applications.

3. Tenth Year Mathematics. The work in this year has traditionally dealt with deductive geometry. Our proposal would continue to stress the deductive method, but would apply it not merely to geometrical subject matter, but to material from algebra and other subjects as well. To achieve this it is not necessary that all of the geometric content of the course be organized into one logical sequence; the role of undefined terms, definitions, assumptions, theorems, can be taught by exhibiting several instances of short groups of propositions. Each such group would illustrate the meaning of a deductive system. Furthermore, it is desirable that non-geometric material—for example, from algebra and other subjects—be organized in this fashion. Our report gives some detailed examples along these lines. In addition to the geometric content of the year's work—and we would urge that the simpler concepts of three dimensional geometry be considered along with their plane counterparts—it is essential that the study and development of the algebra begun in the 9th grade be continued here. This is best done by introducing the subject of analytic geometry at this time and carrying it to a point where the student is able to prove simple theorems by algebraic methods. The equations of simple curves (circle, parabola) can also be introduced. The student will thus continue his use of algebra and at the same time increase his geometric insight. In this connection the study of trigonometry should also be begun.

4. Eleventh Year Mathematics. The work of this year should serve as an introduction to analysis. Since the students taking mathematics at this level are generally those who are planning to go to college, the material should be of a college preparatory character. The subject matter is a continuation of work begun at earlier levels and is essentially algebra, analytic geometry, and trigonometry. Among the subjects to be treated in algebra are the theory of polynomials—including the remainder theorem; systems of linear equations—including determinants; complex numbers; logarithms. The work in analytic geometry should carry the student through the study of linear geometry and the elements of conic sections. The trigonometry to be studied in this year should be largely analytic trigonometry, the work being centered around a study

of the trigonometric functions as functions, with a minimum of work done in geometric trigonometry and computation. Moreover, the application of trigonometric ideas to complex numbers should certainly be included here.

5. Twelfth Year Mathematics. The primary objective of the work in this year is to give a substantial introduction to differential and integral calculus, with enough applications to bring out the meaning and to illustrate the fundamental importance of this subject. The work to be done in the two previous years was planned so as to lead up to this subject, and the necessary prerequisite topics from algebra, analytic geometry, and trigonometry were included. It is expected that skills in these subjects will be further developed when they are employed in the calculus, and that those topics which were not covered during previous years will be studied now. In particular, the analytic geometry previously studied would be used here and would be enriched by applications of the calculus.

6. Remarks. It will be seen that there is nothing very unconventional about the subject matter suggested in our program. In fact, it is quite definitely built around the development and applications of analysis. It seems to the writer that this is the way it should be, especially when one keeps in mind the variety of students for whom this work is planned. A student entering college prepared with such a program will have many advantages. He will in fact be ready to take the normal sophomore course in mathematics. Thus, if he becomes a mathematician he will be able to accelerate his progress in analysis and will, because of this fact, be able to take additional work in other fields, such as higher algebra, advanced geometry, statistics, mechanics, and so forth, and thus round out his mathematical background. If the student is preparing to study science or engineering, he will be greatly helped by having had a course in calculus before he enters college; his first course in physics, for example, can then be a real meaningful introduction to the subject. Finally, there will be certain students who will not need to study mathematics in college, because the course here outlined will give them the kind of preparation that is sufficient for their needs. Such students will thus be free to take work in college which will perhaps be more to their advantage educationally.

As stated at the beginning, the program here outlined was originally proposed by our committee for those exceptional students who wish to present a certain amount of mathematics for advanced credit. It seems to us, however, that the ideas behind it could be equally well applied to secondary school mathematics in general. Thus it may be that work along these lines will become the standard program in secondary school mathematics. Or, it may be desirable for the less able student to stretch out over three years the work that we have laid out for the tenth and eleventh years. As a third possibility, a student may wish to take the tenth and eleventh year courses proposed in our program, and not take the twelfth year course; courses in advanced algebra, statistics, solid

geometry, and so forth could be made available for such a student in his twelfth year. Such students would then take the normal freshman college course when they enter college. There are already many schools in this country in which work of this type in mathematics is being done. It is our hope that this will be continued and that other schools will follow.

ALTERNATIVE SOLUTION TO THE EHRENFEST PROBLEM*

F. G. HESS, University of British Columbia

1. Introduction. Explicit expressions for the probabilities connected with the so-called Ehrenfest model (see Section 2 below) have been obtained by M. Kac [1], who has applied the usual method of dealing with problems involving discrete Markov chains. It is the purpose of this article to show that the solution to the problem can be obtained in a simpler way if the problem is formulated in terms of a direct product representation.† Such a formulation should be useful for calculating probabilities connected with discrete Markov chains of similar complexity.

2. The Ehrenfest model. In order to illustrate certain features of statistical mechanics, P. and T. Ehrenfest [2] have considered the following simple model.

$2R$ labelled balls are placed in 2 boxes. We have $2R$ labelled tickets, a one-to-one correspondence existing between the tickets and balls. A ticket is drawn at random. The ball represented by this ticket is removed from the box it is in and placed in the other box. The ticket is replaced in the pack which is then shuffled. The process is repeated.

One of the questions which arose in connection with this model is—what is the probability, $P(n|m; s)$, that there are $R+m$ balls in box 1 after s draws if there are $R+n$ balls in box 1 initially? This is the problem treated here.

3. Procedure. A brief outline of the procedure for calculating P follows. A set of orthonormal column vectors, ξ_i, is found such that each ξ_i represents a possible state of the system. (We shall define a state explicitly below.) A matrix operator H is then found such that when it operates on a state vector, ξ_i, it forms the sum of all those state vectors which can result from ξ_i in a single draw, *i.e.*,

* I wish to thank Professor W. Opechowski for his interest in this problem. I should also like to express my indebtedness to the National Research Council of Canada for a Studentship.

† The referee has kindly brought to my attention the existence of a paper by A. J. F. Siegert [4], in which essentially the same method is used to obtain the solution to the Ehrenfest problem. However, the present paper differs considerably from that of Siegert's in the explicit formulation of the method. It may be mentioned that Siegert has investigated other problems for which the same method of solution applies.

$$H\xi_i = \sum_j p^1_{ji}\xi_j, \tag{1}$$

where p^1_{ji} is the probability that the state ξ_j is obtained from the state ξ_i in a single draw. From (1) it can be shown that [3],

$$H^s\xi_i = \sum_j p^s_{ji}\xi_j, \tag{2}$$

where

$$p^s_{ji} = \xi_j' H^s \xi_i \tag{3}$$

is the probability that the state ξ_j is obtained after s draws if the system is initially in the state ξ_i; ξ_j' is the transpose of ξ_j. To evaluate p^s_{ji} we must know the eigenvalues and eigenvectors of H and the expansion of ξ_i in terms of these eigenvectors.

Let us now define explicitly a state of the system. Following Kac, we could say that a state is defined if we know the number of balls in box 1. If we let $\xi_m(-R \leqq m \leqq R)$ be a column vector having the number 1 in its $R+m$th row and zeros elsewhere, then ξ_m would represent the state having $R+m$ balls in box 1. By calculating all p^1_{ji} from their definition we then obtain H from (1) as a $(2R+1)$ by $(2R+1)$ matrix. We must then find the eigenvalues and eigenvectors of H in order to calculate $p^s_{ji} = P(i|j;\ s)$. Essentially, this is the procedure carried out by Kac.

However, in this paper we shall introduce a more complicated definition of a state which simplifies the solution of the problem. We shall say that a state is defined if we know which ball is in which box. Thus, we have $\binom{2R}{R+m}$ different states possible having $R+m$ balls in box 1. As a result, let the column vector ξ_{mk} denote the kth state of the possible $\binom{2R}{R+m}$ states which have $R+m$ balls in box 1; k is just an index distinguishing between the different $\binom{2R}{R+m}$ states. Replacing i and j in (1), (2), and (3) by mk and nl respectively, we obtain,

$$p^s_{mk,nl} = \xi'_{mk} H^s \xi_{nl} \qquad (\xi'_{mk}\xi_{nl} = \delta_{mn}\delta_{kl}). \tag{4}$$

Now $\sum_k p^s_{mk,nl}$ is the probability that there are $R+m$ balls in box 1 after s draws if the system is initially in one of the $\binom{2R}{R+n}$ states ξ_{nl} having $R+n$ balls in box 1. This probability must be independent of l. Hence

$$P(n\,|\,m;\ s) = \binom{2R}{R+n}^{-1} \sum_{kl} p^s_{mk,nl}. \tag{5}$$

If we let

$$X(a) = \sum_m a^{R+m} \sum_k \xi_{mk}, \tag{6}$$

where a is some parameter, then $\binom{2R}{R+n} P(n|m;s)$ is the coefficient of $a^{R+m}b^{R+n}$ in the expansion of

$$g(a, b) = X'(a)H^s X(b). \tag{7}$$

In the next section, we shall express ξ_{mk}, $X(a)$, and H explicitly. For our formulation of the problem it is a fairly simple matter to find the eigenvalues and eigenvectors of H. In addition $X(a)$ is easily expressed in terms of the eigenvectors of H.

4. Solution. Suppose we had only 1 ball. Let α, β defined by

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{8}$$

represent the fact that the ball is in box 1, box 2 respectively. If

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \left(S^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \right), \tag{9}$$

then, for the case of only 1 ball, S is the operator H because

$$S\alpha = \beta, \qquad S\beta = \alpha. \tag{10}$$

For the case of $2R$ balls, we let ϵ_i be α or β depending on whether the ith ball is in box 1 or box 2. Then, a state ξ_{mk} of the system can be represented by the direct product of the matrices ϵ_i, *i.e.*,

$$\xi_{mk} = \epsilon_1 \times \epsilon_2 \times \cdots \times \epsilon_{2R}, \tag{11}$$

where $R+m$ of the ϵ_i's are α's and the rest β's. The ξ_{mk}'s are orthonormal.

(It may be noted here that our formulation of the problem is easily extended to include similar problems involving more than 2 boxes.)

If we define

$$[a\alpha + \beta]^{2R} \equiv [a\alpha + \beta] \times [a\alpha + \beta] \times \cdots \times [a\alpha + \beta], \tag{12}$$

then

$$[a\alpha + \beta]^{2R} = \sum_m a^{R+m} \sum_k \xi_{mk},$$

and so, from (6),

$$X(a) = [a\alpha + \beta]^{2R}. \tag{13}$$

H is given by

$$H = \frac{1}{2R} \sum_{i=1}^{2R} S_i$$

where S_i is the direct product of $2R$ matrices, one of which is S (9), located in the ith position, and all the rest of which are the unit matrix, I. Each S_i, operating on some ξ_{mk}, changes the ith ball from the box it is in to the other box as a result of (10). Thus H, operating on ξ_{mk}, produces a sum of $2R$ new states, each one obtained from the state ξ_{mk} by moving one of the balls. $1/2R$ is the probability that a particular ball interchanges boxes in a draw.

Hence,

$$H\xi_{mk} = \sum_{nl} p^1_{nl,mk}\xi_{nl}, \tag{14}$$

each $p^1_{nl,mk}$ being $1/2R$ or zero depending on whether or not ξ_{nl} can be obtained from ξ_{mk} by moving just one ball.

We now evaluate $g(a, b)$ in (7). We first find the eigenvalues and eigenvectors of H. We note that μ and ν defined by

$$\mu = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad \nu = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{15}$$

are eigenvectors of S to the eigenvalues 1 and -1 respectively. Furthermore, it is easily shown that each orthonormal vector η_{mk} defined by

$$[c\mu + \nu]^{2R} = \sum_m c^{R+m} \sum_k \eta_{mk} \tag{16}$$

is an eigenvector of H to the eigenvalue

$$(R + m)1 + (R - m)(-1) = 2m, \tag{17}$$

i.e.,

$$H\eta_{mk} = 2m\eta_{mk}. \tag{18}$$

Now, from (8) and (15), we have

$$\alpha = \frac{1}{\sqrt{2}}(\mu + \nu), \qquad \beta = \frac{1}{\sqrt{2}}(\mu - \nu). \tag{19}$$

Therefore, from (13), (19) and (16), we obtain

$$\begin{aligned} X(a) &= \frac{1}{2^R}\left[(a + 1)\mu + (a - 1)\nu\right]^{2R} \\ &= \frac{1}{2^R} \sum_m (a + 1)^{R+m}(a - 1)^{R-m} \sum_k \eta_{mk}. \end{aligned} \tag{20}$$

Substituting (20) and (18) into (7) and using the fact that the η_{mk}'s are orthonormal, we obtain

$$(21)\quad g(a,b)=\frac{1}{2^{2R}}\sum_{j=-R}^{R}(a+1)^{R+j}(a-1)^{R-j}(b+1)^{R+j}(b-1)^{R-j}\left(\frac{2j}{2R}\right)^{*}\binom{2R}{R+j}.$$

Defining K_l^j by

$$(22)\quad (a+1)^{R+j}(a-1)^{R-j}=\sum_{l=0}^{2R}K_l^j a^l,$$

we obtain finally,

$$(23)\quad \binom{2R}{R+n}P(n\,|\,m;s)=\frac{1}{2^{2R}}\sum_{j=-R}^{R}\left(\frac{j}{R}\right)^{*}\binom{2R}{R+j}K_{R+m}^{j}K_{R+n}^{j}.$$

From (23), we see that $P(n|m;s)$ has the symmetry property

$$(24)\quad \binom{2R}{R+n}P(n\,|\,m;s)=\binom{2R}{R+m}P(m\,|\,n;s),$$

a result which follows quite easily from the physical nature of the problem.

Kac's result, in terms of our notation, is given by

$$(25)\quad P(n\,|\,m;s)=\frac{1}{2^{2R}}\sum_{j=-R}^{R}\left(\frac{j}{R}\right)^{*}K_{R+m}^{j}K_{R+j}^{n}.$$

This result is the same as ours since

$$(26)\quad \binom{2R}{R+j}K_{R+n}^{j}=\binom{2R}{R+n}K_{R+j}^{n},$$

a relation proved as follows:

From(22) we have

$$(27)\quad \sum_{nj}\binom{2R}{R+j}K_{R+n}^{j}a^{R+n}b^{R+j}=\{b(a+1)+a-1\}^{2R}.$$

Also

$$(28)\quad \sum_{nj}\binom{2R}{R+n}K_{R+j}^{n}a^{R+n}b^{R+j}=\{a(b+1)+b-1\}^{2R}.$$

Since the right-hand sides of (27) and (28) are the same, the relation (26) immediately follows.

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MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee,

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

PASCAL HEXAGONS ASSOCIATED WITH A TRIANGLE*

VICTOR THÉBAULT, Tennie, Sarthe, France

The object of this note is to point out the use that can be made of a little-known theorem concerning a Pascal hexagon inscribed in a conic when the conic is taken as the nine point circle of a triangle.

THEOREM. *If A, a, B, b, C, c are six arbitrary points on a conic (C), then the Pascal lines of the inscribed hexagons $AaBbCc$, $AbBcCa$, $AcBaCb$ are concurrent.*

In the inscribed hexagon $AaBbCc$, the pairs of opposite sides Aa and bC, Bb and cA intersect in D, D' on the corresponding Pascal line; in the inscribed hexagon $AbBcCa$, the pairs of opposite sides Bc and aA, Ca and bB intersect in E, E'; and in the inscribed hexagon $AcBaCb$, the pairs of opposite sides Cb and cB, Ac and aC intersect in F, F'. In order to show that the Pascal lines DD', EE', FF' are concurrent in a point S, it suffices to show that triangles DEF and $D'E'F'$ are homological, or, in other words, that the corresponding sides DE and $D'E'$, EF and $E'F'$, FD and $F'D'$ intersect in three collinear points M, N, P. But this is easily seen to be the case, since these three points belong to the Pascal line of the inscribed hexagon $AcBbCa$.

We also observe that triangles aCD and BcD', whose corresponding vertices lie on the concurrent lines aB, Cc, DD', are homological.

In a triangle $T \equiv ABC$, let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB, let A', B', C' be the feet of the corresponding altitudes, and let H be the orthocenter and K the symmedian point. Let N and P be the points of intersection, with lines AB and CA, of parallels to CA and AB through an arbitrary point M on line BC, and let X, Y, Z be the second points of intersection of circle $(C) \equiv MNP$ with the lines BC, CA, AB.

* Translated from the French by Howard Eves.

COROLLARY I. *The points of intersection A, D, E of the opposite sides of the hexagon $MXYPZN$ inscribed in circle (C) are collinear on the symmedian AK of triangle T, and the Pascal lines of the inscribed hexagons $MPYNZX$ and $MNYXZP$ are concurrent on AK.*

It suffices to show that the Pascal line ADE of the inscribed hexagon $MXYPZN$ coincides with the symmedian AK, for the Pascal lines of the other two hexagons are concurrent on the preceding one by our opening theorem. Now it is clear that the line ADE remains fixed as M varies along BC, for triangle DPZ remains homothetic to itself inasmuch as trapezoid $MPZN$ is isosceles and the angles at D, P, Z of the triangle are equal to the angles at C, A, B of T. It follows that the distances of D from sides CA and AB are proportional to these sides, and the proof is complete.

COROLLARY II. *The points A, $a_1 \equiv (A_1B_1,\ C'A')$, $a_1' \equiv (C_1A_1,\ A'B')$ are collinear on the symmedian AK of triangle T; the points B, $(B_1C',\ A'B')$, $(B'C_1,\ A_1B_1)$ are collinear on a line (D); the points C, $(B'C_1,\ C'A')$, $(B_1C',\ C_1A_1)$ are collinear on a line (D'); the lines (D), (D') intersect on AK.*

This proposition is a particular case of the preceding one, the circle (C) being the nine point circle of triangle T. The symmedian AK coincides with the Pascal line of the inscribed hexagon $B'B_1A_1C_1C'A'$, and (D), (D') coincide with the Pascal lines of the hexagons $B'C_1A_1A'C'B_1$, $B'A'A_1B_1C'C_1$, obtained by permuting the vertices of the first hexagon according to the statement of the opening theorem.

Alternatively, one can show that AK passes through a_1 and a_1' by observing that these points bisect the segments CC_2, BB_2 of parallels to $B'C'$ drawn through C, B and limited by AB, AC. In fact, the parallel to AB drawn through C cuts $C'A'$ and $C'B'$ at the vertices V, W of an isosceles triangle $C'VW$ of altitude $C'C$. Consequently, $C'A'$ cuts CC_2 in its midpoint, which is situated on A_1B_1 and is thus located at a_1. A similar conclusion follows for a_1' on BB_2, whence the points A, K, a_1, a_1' are collinear.

The point of intersection C_2' of $A'B'$ and CC_2 is collinear with points H and B_2 on a symmedian of triangle HBC.

COROLLARY III. *The sets of points*

$$(C_1C',\ B'B_1), \quad (C'A_1,\ B_1A'), \quad (A_1B',\ A'C_1),$$
$$(A_1A',\ C'C_1), \quad (A'B_1,\ C_1B'), \quad (B_1C',\ B'A_1),$$
$$(B_1B',\ A'A_1), \quad (B'C_1,\ A_1C'), \quad (C_1A',\ C'B_1),$$

determine three concurrent cevians (a), (b), (c) of triangle T.

The lines (a), (b), (c) coincide with the Pascal lines of the inscribed hexagons $C_1C'A_1B'B_1A'$, $C_1B'A_1A'B_1C'$, $C_1A'A_1C'B_1B'$, and are concurrent by virtue of the opening theorem. The point of concurrency has $(a^2 \cos A,\ b^2 \cos B,\ c^2 \cos C)$ for trilinear coordinates with respect to the fundamental triangle T.

COROLLARY IV. *The Pascal lines of the inscribed hexagons* $C'C_1A_1A'B_1B'$, $C'A'A_1B'B_1C_1$, $C_1C'A'A_1B_1B'$ *form a triangle homothetic to the fundamental triangle* T.

For the opposite sides B_1C_1 and A_1A', C_1A_1 and B_1B', A_1B_1 and C_1C' of these hexagons are parallel.

With the six points A_1, B_1, C_1, A', B', C' one can form 60 hexagons inscribed in the nine point circle of triangle T. The consideration of the 60 corresponding Pascal lines, which are not necessarily all distinct,* and the points which they determine by their intersections in triples according to the statement of the opening theorem of this note, would constitute a subject for study.

* Ch. Bioche, *Bulletin de la Société Mathématique de France*, t. 58, p. 27, (1930).

A NOTE ON NORMAL MATRICES

W. V. PARKER, Alabama Polytechnic Institute

A matrix A is said to be normal if $AA^*=A^*A$, where A^* denotes the transpose conjugate of A [1]. The purpose of this note is to extend a theorem of an earlier paper [2] concerning such matrices. That theorem may be stated as follows:

THEOREM 1. *If* $A=(a_{ij})$ *is an* $n\times n$ *normal matrix and the characteristic roots of* A *are* $a_{11}, a_{22}, \cdots, a_{nn}$ *then* A *is a diagonal matrix.*

In this note we establish

THEOREM 2. *If* $A=(a_{ij})$ *is an* $n\times n$ *normal matrix which can be partitioned into* $A=(A_{\alpha\beta})$, $\alpha, \beta=1, 2, \cdots, k$, *such that the characteristic function of* A *is the product of the characteristic functions of the square matrices* $A_{\alpha\alpha}$, $\alpha=1, 2, \cdots, k$, *then* $A=\text{diag.}\{A_{11}, A_{22}, \cdots, A_{kk}\}$.

We first prove this for A an Hermitian matrix.

LEMMA 1. *Let* $A=(A_{\alpha\beta})$, $\alpha, \beta=1, 2, \cdots, k$, *be an* $n\times n$ *Hermitian matrix and let* $D=\text{diag.}\{A_{11}, A_{22}, \cdots, A_{kk}\}$. *If* A *and* D *have the same characteristic function, then* $A=D$.

The coefficient of x^{n-2} in the characteristic function of an $n\times n$ matrix is the sum of all second order principal minors of the matrix. Denote these coefficients for matrices A and D by S and S' respectively. Then $S-S'=\sum a_{ij}\bar{a}_{ij}$ where the sum is taken over all i, j such that $i<j$ and no $A_{\alpha\alpha}$ contains both the ith row and the jth row of A. Since $S=S'$, $a_{ij}=0$ for every element of every matrix $A_{\alpha\beta}$, $\alpha\neq\beta$. This completes the proof of Lemma 1.

LEMMA 2. *Let* $A=(A_{\alpha\beta})$, $\alpha, \beta=1, 2, \cdots, k$, *be an* $n\times n$ *normal matrix and let* $D=\text{diag.}\{A_{11}, A_{22}, \cdots, A_{kk}\}$. *If* A *and* D *have the same characteristic function,* D *is also normal.*

There exist unitary matrices U_α such that $U_\alpha A_{\alpha\alpha} U_\alpha^*$, $\alpha = 1, 2, \cdots, k$, is triangular. Let $U = \text{diag.}\{U_1, U_2, \cdots, U_k\}$. Then UAU^* is a normal matrix having its characteristic roots as diagonal elements. It follows from Theorem 1 that UAU^* is a diagonal matrix and consequently $U_\alpha A_{\alpha\alpha} A_\alpha^*$ is a diagonal matrix and $A_{\alpha\alpha}$ is normal. A direct sum of normal matrices is a normal matrix and hence D is normal.

We now return to the proof of Theorem 2. We may write $A = G + iH$, where G and H are Hermitian. Also, write $G = (G_{\alpha\beta})$ and $H = (H_{\alpha\beta})$ where $G_{\alpha\beta} = \frac{1}{2}(A_{\alpha\beta} + A_{\alpha\beta}^*)$ and $H_{\alpha\beta} = (A_{\alpha\beta} + A_{\beta\alpha}^*)/2i$. Since A is normal there exists a unitary matrix V such that $VAV^* = \text{diag.}\{L_1, L_2, \cdots, L_k\}$ where L_α is a diagonal matrix having as diagonal elements the characteristic roots of $A_{\alpha\alpha}$. From Lemma 2, we know that $A_{\alpha\alpha}$ is normal and hence there exist unitary matrices U_α such that $U_\alpha A_{\alpha\alpha} U_\alpha^* = L_\alpha$, $\alpha = 1, 2, \cdots, k$. If $U = \text{diag.}\{U_1, U_2, \cdots, U_k\}$ and $W = U^*V$, then $WAW^* = \text{diag.}\{A_{11}, A_{22}, \cdots, A_{kk}\}$; consequently $WGW^* = \text{diag.}\{G_{11}, G_{22}, \cdots, G_{kk}\}$, and $WHW^* = \text{diag.}\{H_{11}, H_{22}, \cdots, H_{kk}\}$. It then follows from Lemma 1 that $G = \text{diag.}\{G_{11}, G_{22}, \cdots, G_{kk}\}$ and $H = \text{diag.}\{H_{11}, H_{22}, \cdots, H_{kk}\}$. Thus we have $A = \text{diag.}\{A_{11}, A_{22}, \cdots, A_{kk}\}$.

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2. W. V. Parker, Characteristic roots and the field of values of a matrix, Duke Math. Jour., vol. 15, 1948, pp. 439–442.

A NOTE ON A GENERALIZATION OF THE CAUCHY-MACLAURIN INTEGRAL TEST

T. A. NEWTON, University of Nebraska

1. De la Vallee-Poussin presented the following theorem as an exercise [8, p. 414]. (We shall use the notation $\Delta M_n = M_{n+1} - M_n$.)

THEOREM 1. *If $0 < M_n \leqq M_{n+1}$, $n = 1, 2, 3, \cdots$, $M_n \to \infty$ with n, and if $F(x)$ is positive monotone decreasing, then $\sum_{n=1}^{\infty} F(M_{n+1})\Delta M_n$ converges with $\int^{\infty} F(x)dx$, and $\sum_{n=1}^{\infty} F(M_n)\,\Delta M_n$ diverges with $\int^{\infty} F(x)dx$.*

When added conditions are placed on $\{M_n\}$, the following theorem results.

THEOREM 1a. *If, in Theorem 1, either $\Delta M_n = O(1)$ or $M_{n+1} = O(M_n)$, then $\sum_{n=1}^{\infty} F(M_{n+1})\Delta M_n$ and $\sum_{n=1}^{\infty} F(M_n)\Delta M_n$ converge and diverge together.*

These results have been known for some time. But since the author has been unable to find explicit proof of both theorems in the more readily available literature, they are proved in Section 2.

It is seen that these results contain the Cauchy-Maclaurin integral test and Cauchy's condensation test as special cases. By setting $F(x) = 1/x^p$, the Abel-Dini theorem is easily proved using these results [6]. In turn, the Abel-

Dini theorem is closely related to Kummer's very general criteria [4, p. 310]. Extensive discussions concerning convergence and divergence criteria based upon Theorem 1 and 1a are given by Rajagopal in [5] and [7].

In Sections 3 and 4 of this note, we shall show that aside from serving as a basis for various convergence and divergence criteria, Theorems 1 and 1a give us a means for examining the behavior of the general terms of certain convergent series.

2. *Proof of Theorem* 1. Since $F(x)$ is monotone decreasing and $\{M_n\}$ is monotone increasing, it follows that $F(M_v) \geqq F(x) \geqq F(M_{v+1})$ for $M_v \leqq x \leqq M_{v+1}$, $v=1, 2, \cdots$. Consequently,

$$F(M_v)\Delta M_v \geqq \int_{M_v}^{M_{v+1}} F(x)dx \geqq F(M_{v+1})\Delta M_v$$

for $v=1, 2, 3, \cdots$. Summing for $v=1, 2, \cdots n$, it follows that

$$\sum_{v=1}^{n} F(M_v)\Delta M_v \geqq \int_{M_1}^{M_{n+1}} F(x)dx \geqq \sum_{v=1}^{n} F(M_{v+1})\Delta M_v$$

for $n>0$. Letting n become infinite, we obtain the desired result.

Proof of Theorem 1a. Considering first that $\Delta M_n = O(1)$, determine G such that $\Delta M_n \leqq G$ for all n. It follows that

$$0 \leqq \left\{ \sum_{v=1}^{n} F(M_v)\Delta M_v - \sum_{v=1}^{n} F(M_{v+1})\Delta M_v \right\} \leqq G \sum_{v=1}^{n} \{F(M_v) - F(M_{v+1})\}$$

or

$$0 \leqq \left\{ \sum_{v=1}^{n} F(M_v)\Delta M_v - \sum_{v=1}^{n} F(M_{v+1})\Delta M_v \right\} \leqq GF(M_1)$$

for $n>0$. Letting n become infinite, it follows that since the terms of both series are positive then the series must either both diverge or both converge.

For the case in which $M_{n+1} = O(M_n)$, we first note that if $\{M_n\}$ satisfies the hypothesis of Theorem 1, then so does $\{KM_n\}$ for any $K>0$. It then follows from Theorem 1 that the divergence of $\int^{\infty} F(x)dx$ implies the divergence of both $\sum_{n=1}^{\infty} F(M_n)\Delta M_n$ and $\sum_{n=1}^{\infty} F(KM_n)\Delta(KM_n)$. But since $M_{n+1} = O(M_n)$, we can choose K such that $M_{n+1} \leqq KM_n$ for $n>0$, and it follows from the monotone nature of $F(x)$ that $K^{-1}F(KM_n)\Delta(KM_n) \leqq F(M_{n+1})\Delta M_n$ for $n>0$. It follows by comparison that $\sum_{n=1}^{\infty} F(M_{n+1})\Delta M_n$ diverges with $\int^{\infty} F(x)dx$. That the series $\sum_{n=1}^{\infty} F(M_n)\Delta M_n$ and $\sum_{n=1}^{\infty} F(M_{n+1})\Delta M_n$ converge with $\int^{\infty} F(x)dx$ follows in a similar manner from the inequality $F(M_n)\Delta M_n \leqq KF(K^{-1}M_{n+1})\Delta(K^{-1}M_n)$ for $n>0$.

3. For the remainder of this paper, $\sum c_n$ (or $\sum c(n)$) and $\sum d_n$ (or $\sum d(n)$) denote respectively arbitrary convergent and divergent series of positive terms,

and $\{M_n\}$ denotes an arbitrary sequence satisfying the hypothesis of Theorem 1. The following facts have been pointed out in classical literature.

(3.1) If $\sum c_n$ is monotone, then $nc_n \to 0$, and if $p_n \to +\infty$ then there exists a monotone $\sum c_n'$ such that $\overline{\lim}\, np_n c_n' = +\infty$.

(3.2) Given any $\sum c_n$, there always exists a monotone $\sum d_n$ such that $d_n \to 0$ and $\underline{\lim}\, d_n/c_n = 0$.

(3.3) Given any $\sum d_n$ where $d_n \to 0$, there always exists a $\sum c_n$ such that $\overline{\lim}\, c_n/d_n = +\infty$.

These results are discussed by Knopp [4, pp. 300–305]. More recently, (3.1) was extended by R. W. Hamming in [2]. His results are included in the following theorem.

THEOREM 2. *If $\sum c(n)$ is monotone, M_{n+1}/M_n bounded from 1 and ∞, and $\sum c(M_n)\Delta M_n$ monotone decreasing, then $n(\log n)c(n) \to 0$.*

Proof. Since $M_{n+1} = O(M_n)$, it follows from Theorems 1 and 1a that $\sum c(M_n)\Delta M_n$ is convergent. The remainder of the proof is quite similar to that used by Hamming in [2], where he considers the particular case $M_n = 2^n$, and will not be presented here.

Furthermore, Theorem 2 is the best such statement in that if $p_n \to +\infty$, then there exists a $\sum c(n)$ and $\{M_n\}$ satisfying the conditions of Theorem 2 such that $np_n (\log n)c(n)$ does not approach zero. A theorem by Hamming in [2] suffices as a proof of this assertion in that he proved that if we let $M_n = 2^n$, then there exists a monotone $\sum c(n)$ such that $\sum 2^n c(2^n)$ is also monotone and yet $np_n (\log n)c(n)$ does not approach zero.

4. With respect to statements (3.2) and (3.3), Hamming in [3] and Aryeh Dvoretzky in [1] have proved some theorems which show, in a certain sense, the rarity of the occurrence $c_n \geqq d_n$ for arbitrary monotone $\sum c_n$ and $\sum d_n$. Theorem 1 of Dvoretzky implies Hamming's results, while the following theorem implies Theorem 1 of Dvoretzky. This theorem is only a slight extension of Dvoretzky's theorem, but is thought to be of interest since its proof is based upon Theorems 1 and 1a above.

THEOREM 3. *If $\sum c_n$ and $\sum d_n$ are monotone, and for arbitrary $\{M_n\}$, either $\Delta M_n = O(1)$ or $M_{n+1} = O(M_n)$, then there exist infinitely many n such that for all v with $M_n \leqq v \leqq M_{n+1}$, we have $c_v < d_v$.*

Proof. Assume that $c(x)$ and $d(x)$ are monotone decreasing functions such that $c(n) = c_n$, and $d(n) = d_n$. Further assume that there exists an n_0 such that for all $n > n_0$, $c(M_n) \geqq d(M_{n+1})$, or $c(M_n)\Delta M_n \geqq d(M_{n+1})\Delta M_n$ where $\{M_n\}$ satisfies the hypothesis of the theorem. But it follows from Theorems 1 and 1a that $\sum d(M_{n+1})\Delta M_n$ diverges with $\sum d_n$. Consequently the latter inequality implies the divergence of $\sum c(M_n)\Delta M_n$. Again, considering Theorems 1 and 1a, we see that the divergence of $\sum c(M_n)\Delta M_n$ implies the divergence of $\sum c_n$,

thus arriving at a contradiction. Consequently there exist infinitely many n such that $c(M_n) < d(M_{n+1})$. However, due to the monotone nature of the functions $c(x)$ and $d(x)$, if the latter inequality is true for n, then for all v with $M_n \leqq v \leqq M_{n+1}$, $c(v) \leqq c(M_n) < d(M_{n+1}) \leqq d(v)$.

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TRISECTION APPROXIMATION

FREE JAMISON, San Jose State College, California

On January 13, 1953, C. R. Lindberg† proposed a method for the approximate trisection of the angle. His method with one slight refinement follows. Using V, the vertex of the angle as a center, let a circle intersect the sides at A and B (Fig. 1). Draw BV and extend to C on the circle. Through D, the midpoint of $\widehat{AB}$, draw CD and extend to E so that $DE=BC$. Draw EV which makes $\angle BVE$ approximately $1/3\angle BVA$. The error can be found by consulting a tables of sines, for in $\triangle EVD$, $\sin V = 2 \sin E$, and $\angle V + \angle E = 1/4 \angle BVA$. By solving, the error for a given positive angle $X < \pi$ is

$$X/12 - \arctan\left[(\sin X/4)/(2 + \cos X/4)\right].$$

If $Y = X/4$, the error is

$$\frac{Y}{3} - \arctan\frac{\sin Y}{2 + \cos Y} < \frac{Y}{3} - \arctan\frac{Y - Y^3/6}{3 - Y^2/2 + Y^4/24}$$

$$< \frac{Y}{3} - \left[\frac{Y}{3} - \frac{Y^5}{3(72 - 12Y^2 + Y^4)}\right]$$

$$+ 1/3\left(\frac{Y}{3}\right)^5 < \frac{Y^5}{194} + \frac{Y^5}{81} < \frac{X^5}{4000}.$$

† C. R. Lindberg: Method of trisecting an angle, Unpublished paper, San Jose State College.

In an extension of Lindberg's method (Fig. 2), D is located so that $\widehat{BD} = (3/8)\widehat{BA}$ and C is diametrically opposite F where $\widehat{BF} = (1/4)\widehat{BA}$. In this extension the error for a given positive angle $X < \pi$ is

$$X/48 - \arctan\left[(\sin X/16)/(2+\cos X/16)\right] < X^5/320{,}000.$$

A comparison of the accuracy and simplicity of the five approximations described by Yates‡ with those based on Lindberg's method is given in the following table. The column headed "N" is the total of the number of arcs, number of settings of compasses to a definite length, and the number of lines required including an extension of a side where necessary.

Method	N	Approximate Maximum Errors	
		Acute Angles	Obtuse Angles
Von Cusa—Snellius	15	3° 26′	*
Dürer	>20	18″	31′
Karajordanoff	16	2′ 12″	*
Kopf-Perron	14	15″	*
D'Ocagne	8	22′	*
Lindberg	9	3′	22′
Lindberg's Extended	12	3″	20″

* Not applicable or else errors are more than 3°.

Topel§ made some improvement on the method of Kopf-Perron applicable to angles less than 45°, which reduced the error to less than 0.5″. However, Lindberg's method extended requires a smaller N than Topel's and gives an approximate maximum error of 0.3″ for angles less than 45°.

The small errors by Lindberg's method are not surprising when it is considered that small angles are almost proportional to their sines; and a triangle with two small angles whose opposite sides are in the ratio 2:1 will have its smallest angle approximately equal to one-third of the exterior angle at the third vertex. It may be of interest that Lindberg arrived at his method unaware

‡ Yates, R. C.: The Trisection Problem, West Point, N. Y.,

§ Topel, Bernard J.: Concerning a remark of Canon Lemaître about Kopf's trisection of the angle, Reports of a Mathematical Colloquium, Second Series, Issue 1, Notre Dame, 1939, pp. 49–52.

that $DE=2VD$; the refinement mentioned heretofore concerned the method of locating but not the position of E.

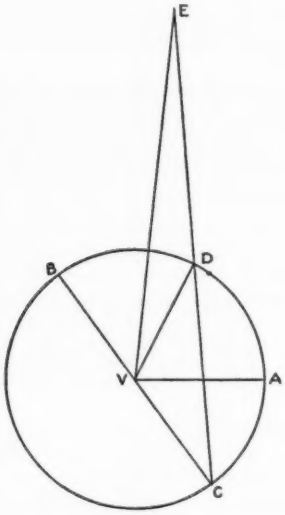


FIG. 1

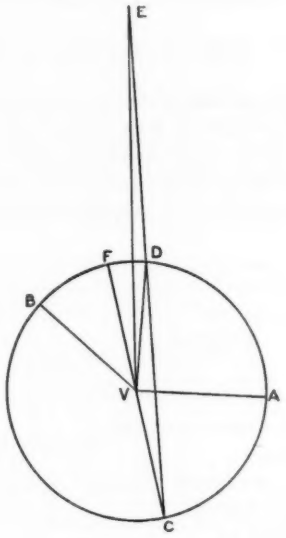


FIG. 2

DIVISORS OF ZERO IN POLYNOMIAL RINGS

W. R. SCOTT, The University of Kansas

The purpose of this note is to give a brief proof of the following theorem, proved by Alexandra Forsythe in a note with the above title, this MONTHLY, vol. 50, 1943, pp. 7–8. (*Cf.* also, N. H. McCoy, *Rings and Ideals*, p. 34, Th. 4.)

THEOREM. *Let R be a commutative ring, and let $R[x]$ be the ring of polynomials over R. If f in $R[x]$ is a divisor of zero, then there is a c in R such that $c\neq 0$, $cf=0$.*

Proof. Deny the theorem, and let g be a non-zero polynomial of smallest degree such that $fg=0$. Let

$$f = a_0 + a_1x + \cdots + a_mx^m,$$
$$g = b_0 + b_1x + \cdots + b_nx^n,$$

where $b_n\neq 0$ and $n\geqq 1$. Since $b_nf\neq 0$, $a_ib_n\neq 0$ for some i, and therefore $a_ig\neq 0$. Let r be the largest integer such that $a_rg\neq 0$. Then

$$fg = (a_0 + \cdots + a_rx^r)(b_0 + \cdots + b_nx^n) = 0.$$

Hence $a_rb_n=0$ and deg $(a_rg)<n$. However, $(a_rg)f=0$, which is a contradiction.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

NOTE ON HERO'S FORMULA

J. P. BALLANTINE, University of Washington

In the development of the formulas for the solution of a triangle, given three sides, by logarithms, the usual derivation of the formula for r or K (the area) is rather long, starting from the law of cosines. Following is a much briefer derivation.

The formulas, $\tan \frac{1}{2}A = r/(s-a)$, $\tan \frac{1}{2}B = r/(s-b)$, and $\tan \frac{1}{2}C = r/(s-c)$ can be read easily from the diagram. Then

$$\tfrac{1}{2}A + \tfrac{1}{2}B = 90^\circ - \tfrac{1}{2}C.$$

Take the tangent of both members:

$$\frac{\tan \frac{1}{2}A + \tan \frac{1}{2}B}{1 - \tan \frac{1}{2}A \tan \frac{1}{2}B} = \frac{1}{\tan \frac{1}{2}C}.$$

Clear:

$$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}C + \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C = 1 - \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B,$$
$$\tan \tfrac{1}{2}B \tan \tfrac{1}{2}C + \tan \tfrac{1}{2}A \tan \tfrac{1}{2}C + \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B = 1.$$

Now substitute the values of the various tangents:

$$\frac{r^2}{(s-b)(s-c)} + \frac{r^2}{(s-a)(s-c)} + \frac{r^2}{(s-a)(s-b)} = 1,$$
$$r^2(s-a+s-b+s-c) = (s-a)(s-b)(s-c),$$
$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

TRIGONOMETRY FROM DIFFERENTIAL EQUATIONS

D. E. RICHMOND, Williams College

1. Introduction. This note shows how analytic trigonometry may be developed in an elementary manner (with no use of infinite series) from the differential equation

$$\text{(I)} \qquad \frac{d^2y}{dt^2} + y = 0$$

to which one is naturally led through the study of simple harmonic motion.

First write (I) as

$$\frac{dx}{dt} = -y \tag{1}$$

by setting

$$\frac{dy}{dt} = x. \tag{2}$$

Multiplying (1) by x and (2) by y and adding,

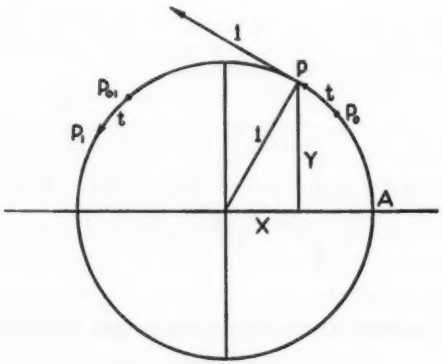
$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0.$$

Hence $x^2+y^2=r^2$ where r is a constant. For given t, x and y are the coordinates of a point on a circle, omitting the trivial case $r=0$.

The radius of this circle becomes 1 if we set $x=rX$, $y=rY$. Then

$$\frac{dX}{dt} = -Y \tag{3}$$

$$\frac{dY}{dt} = X \tag{4}$$



where $X^2+Y^2=1$. Solutions $Y(t)$ so obtained will be said to be *normalized*.

Clearly (3) and (4) give the components of a motion with the uniform velocity 1 in a counterclockwise direction along the unit circle C. If arc length s is measured along C in this direction, $ds/dt=1$ and $s=t+c$. In fact, $s=t$ if s is measured from P_0, the position at $t=0$.

Two normalized solutions of (I), $Y(t)$ and $Y_1(t)$, can differ only in the positions of their initial points, P_0 and P_{01} respectively. But then $Y_1(t)=Y(t+a)$

where a is the arc length from P_0 to P_{01}. Moreover, given one normalized solution of (I), $Y(t)$, then all functions $Y(t+a)$ with constant a are normalized solutions of (I). The general solution of (I) is therefore

$$y = rY(t + a)$$

with arbitrary constants r and a.

Let us define $\sin t$ to be that solution $Y(t)$ for which $Y=0$ and $X(=dY/dt)=1$ at $t=0$, that is, that solution for which $P_0=A$ (see Figure). The corresponding $X(t)=dY(t)/dt$ will be defined to be $\cos t$. From (3) and (4)

$$\frac{d \sin t}{dt} = \cos t,$$

$$\frac{d \cos t}{dt} = -\sin t.$$

It follows at once that $\cos t$ (as well as $\sin t$) is a solution of (I).

It is geometrically obvious that $\sin (t+2\pi)=\sin t$ and $\cos (t+2\pi)=\cos t$ for all t, and also that

$$\sin (-t) = -\sin t, \qquad \cos (-t) = \cos t.$$

The radian (or circular) measure of angles becomes extremely natural.

2. The addition formulas. To derive the addition formulas, we verify by differentiation that

$$c_1 \sin t + c_2 \cos t$$

is a solution of (I) for arbitrary constants c_1 and c_2. Then for some r and some a,

$$c_1 \sin t + c_2 \cos t = r \sin (t + a). \tag{5}$$

Differentiating,

$$c_1 \cos t - c_2 \sin t = r \cos (t + a). \tag{6}$$

Substituting $t=0$ in (5) and (6),

$$c_1 = r \cos a, \qquad c_2 = r \sin a.$$

Inserting these values in (5) and (6) and cancelling the factor r,

$$\sin (t + a) = \sin t \cos a + \cos t \sin a, \tag{7}$$

$$\cos (t + a) = \cos t \cos a - \sin t \sin a. \tag{8}$$

The remaining formulas of analytic trigonometry follow without difficulty.

3. Calculation of $\sin t$ and $\cos t$. If it is desired to *calculate* $\sin t$ and $\cos t$, it suffices to consider the case $t>0$ and integrate both members of $\cos t \leqq 1$ from 0 to t, obtaining $\sin t < t$. Continuing,

$$1 - \cos t < \frac{t^2}{2}$$

or

$$\cos t > 1 - \frac{t^2}{2};$$

$$\sin t > t - \frac{t^3}{3!};$$

$$\cos t < 1 - \frac{t^2}{2!} + \frac{t^4}{4!};$$

. . . . . .

The theorem used here is the intuitively obvious one that if $f(t)$ and $g(t)$ are two different integrable functions such that $f(t) \leqq g(t)$, $(t>0)$, then

$$\int_0^t f(t)dt \leqq \int_0^t g(t)dt, \qquad (t > 0).$$

It is hoped that this note will nourish the suspicion that the conventional semester course in trigonometry involves considerable educational waste.

A GEOMETRIC DETERMINATION OF THE NATURE OF THE ROOTS OF THE CUBIC, BIQUADRATIC, AND QUINTIC EQUATIONS

M. S. KLAMKIN, Polytechnic Institute of Brooklyn

The cubic equation,

$$ax^3 + 3bx^2 + 3cx + d = 0, \qquad a > 0, \tag{1}$$

can be reduced by the substitution, $x = y/m - b/a$, to the canonical form:

$$y^3 + 3py + 2 = 0. \tag{2}$$

Here,

$$m^3[2b^3 - 3abc + da^2] = 2a^3$$
$$m^2[ac - b^2] = pa^2. \tag{3}$$

We will assume that $2b^3 - 3abc + da^2 \neq 0$, and that $ac - b^2 \neq 0$, for otherwise $x = -b/a$ is a root or the cubic can be solved by completing the cube.

Let us now consider the graphs of

$$z = y^2 + 2/y, \quad \text{and} \quad z = -3p. \tag{4}$$

The first curve is asymptotic to $y=0$ and to the parabola $z=y^2$. Also, the curve has a minimum point at $(1, 3)$ and an inflection point at $(-2^{1/3}, 0)$. The second curve is a straight line parallel to the y-axis. The intersections of these two

curves represent the real roots of equation (2). The possible intersections for different values of p are shown in Figure 1.

Consequently, if $-p>1$, there are three real and unequal roots. If $-p=1$, there are three real roots, two of which are equal. And, finally, if $-p<1$, there is only one real root, and the other two are complex.

The nature of the roots is also determined by the sign of the discriminant. However, the two criteria are equivalent, since here

$$\Delta = 4(1 + p^3).$$

This method not only determines the nature of the roots but affords a convenient way of obtaining the real roots of a cubic, since the curve of Figure 1

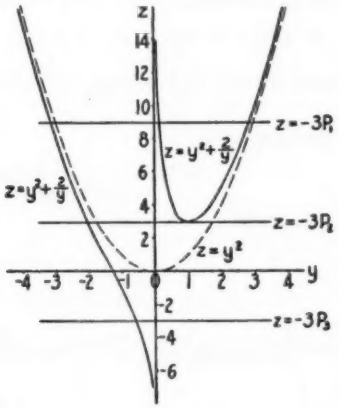


FIG. 1

is the same for all cubics. It should be noted that for a certain range of p, there is no need to resort to a graphical solution of the equation $y^3+3py+2=0$, since the roots are tabulated in Jahnke, Emde, *Tables of Functions*.

The biquadratic equation,

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0, \qquad a > 0, \tag{5}$$

can be reduced by the substitution, $x=ry-b/a$, to the form

$$y^4 + py^2 + my + n = 0, \tag{6}$$

where $p=1$, -1, or 0, according to whether $ac-b^2>0$, <0, or $=0$, respectively.

Let us now consider the graphs of

$$z = y^4 + py^2, \quad \text{and} \quad -z = my + n. \tag{7}$$

Some possible intersections are shown in Figures 2 and 3. The figure for $p=0$ is similar to that for $p=1$. Thus, when $ac-b^2\geqq 0$, there are either two or zero real

roots. Moreover, if $n \leqq 0$, there are two real roots. When $ac-b^2<0$, there are either four, two, or zero real roots. If in addition, $n<-\frac{1}{12}$, $-\frac{1}{12}<n<\frac{1}{4}$, or $n>\frac{1}{4}$, then there are two, four or two, or two or zero, real roots, respectively. For a more complete description of the roots, we would have to consider the case where the curves of (7) are tangent to each other. This can be done, but not readily. However, by plotting the curves of (7) we can determine all the real roots of (6). The three quartic curves $z=y^4+py^2$ can be drawn once to serve for all quartics, while the straight line $-z=my+n$ is easily drawn.

The quintic equation,

$$ax^5 + 5bx^4 + 10cx^3 + 10dx^2 + 5ex + f = 0, \qquad a > 0, \tag{8}$$

can be reduced by the substitution, $x=sy-b/a$, to the form

$$y^5 + py^3 + my^2 + ny + r = 0, \tag{9}$$

where $p=1$, -1, or 0, according to whether $ac-b^2>0$, <0, or $=0$, respectively.

Let us now consider the graphs of

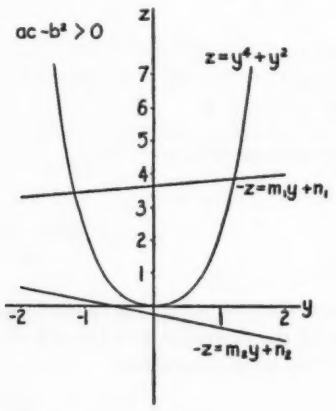


FIG. 2

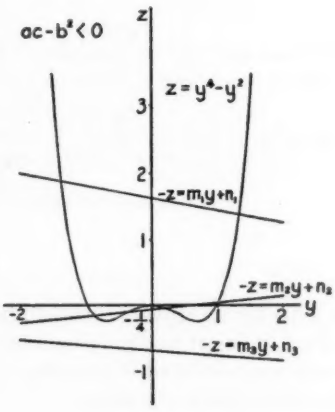


FIG. 3

$$z = y^5 + py^3, \quad \text{and} \quad -z = my^2 + ny + r. \tag{10}$$

The figures are not drawn here, but can easily be constructed by the reader. From them we find that when $ac-b^2 \geqq 0$, there are either three or one real roots, depending on where the vertex of the parabola of (10) is located, and on the sign of m. When $ac-b^2<0$, there are either five, three, or one real roots. Again, by plotting the curves of (10) (the only variable one being the parabola) we can obtain the real roots of (9).

The usefulness of the above method for solving quintic equations would be greatly increased if there existed a simple device for drawing parabolas.

A METHOD FOR THE SOLUTION OF THE GENERAL QUARTIC

T. D. NAGLE, University of Bridgeport

Of general interest is our solution of the general quartic that has a marked similarity to Ferrari's method.

Consider the general quartic

$$f(x) = x^4 + bx^3 + cx^2 + dx + e, \qquad e \neq 0. \tag{1}$$

Case I

Assume

$$d^2 + b^2e - 4ce = 0.*$$

It follows that

$$x^2 + \frac{bx}{2} = \pm\left(\frac{b^2}{4} - c\right)^{1/2}\left(x + \frac{2d}{(4c - b^2)}\right)$$

and the roots of $f(x) = 0$ can be ascertained.

Case II

Assume

$$d^2 + b^2e - 4ce \neq 0.$$

We shall let $x = z + a - b/4$ and obtain by the Taylor expansion,

$$f(x) = f\left(z + a - \frac{b}{4}\right) = z^4 + Bz^3 + Cz^2 + Dz + E = 0. \tag{2}$$

We find

$$z^2 + \frac{B}{2}z = \pm\left(\frac{B^2}{4} - C\right)^{1/2}\left(z + \frac{2D}{(4C - B^2)}\right)$$

if

$$D^2 + B^2E - 4CE = 0.$$

The condition $D^2 + B^2E - 4CE = 0$ is found to be equivalent to the equation

$$a^6 + ra^4 + sa^2 + t = 0, \tag{3}$$

where r, s, and t are functions of b, c, d, and e. Hence, we can employ a root of (3) to reduce the problem to Case I.

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* See L. W. Griffiths, Introduction to the Theory of Equations, p. 33.

PRODUCT OF DETERMINANTS BY INDUCTION

C. M. FULTON, University of California, Davis

In this paper we prove the multiplication theorem of determinants by mathematical induction. We use only properties of determinants which are usually encountered in college algebra texts.

Subscripts take the values $1, \cdots, n$. If a_{ij} is the element of a determinant of order n in the ith row and jth column, its cofactor is denoted by A_{ij} and the value of the determinant by A.

Let a_{ij} and b_{ij} be the elements of two determinants. Consider a third determinant whose elements are defined by the relation

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}.$$

We want to show that $C = AB$.

We need not comment on the validity of the theorem for $n = 1$. Our induction proof will be carried out for the special case in which

$$a_{21} = a_{31} = \cdots = a_{n1} = 0.$$

To save space we suggest that the reader write out the three determinants. In the particular case under consideration, for $i \neq 1$,

$$c_{ik} = a_{i2}b_{2k} + \cdots + a_{in}b_{nk}.$$

The sum on the right side has $n-1$ terms only. We now direct our attention to the cofactors C_{1j}. If our theorem is true for determinants of order $n-1$, these cofactors can be written as products as follows

$$C_{11} = A_{11}B_{11},\ C_{12} = A_{11}B_{12}, \cdots,\ C_{1n} = A_{11}B_{1n}.$$

We then see from the well-known properties of cofactors that

$$\begin{aligned} C &= C_{11}c_{11} + C_{12}c_{12} + \cdots + C_{1n}c_{1n} \\ &= A_{11}B_{11}(a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) \\ &\quad + A_{11}B_{12}(a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2}) + \cdots \\ &\quad + A_{11}B_{1n}(a_{11}b_{1n} + a_{12}b_{2n} + \cdots + a_{1n}b_{nn}) \\ &= A_{11}a_{11}(B_{11}b_{11} + B_{12}b_{12} + \cdots + B_{1n}b_{1n}) \\ &\quad + A_{11}a_{12}(B_{11}b_{21} + B_{12}b_{22} + \cdots + B_{1n}b_{2n}) + \cdots \\ &\quad + A_{11}a_{1n}(B_{11}b_{n1} + B_{12}b_{n2} + \cdots + B_{1n}b_{nn}) = AB. \end{aligned}$$

To complete the proof of our theorem we reduce the general case to the special case that has been treated. With regard to the elements in the first column of the determinant of the a's we distinguish between two cases. Either all of them are zero or at least one is not zero. The first case has been covered. In the second case we may assume without loss of generality that $a_{11} \neq 0$. We then add to the elements of the ith row ($i \neq 1$) the corresponding elements of

the first row multiplied by the same number r_i. This leaves A unchanged. But at the same time C remains unchanged since

$$(a_{i1} + r_i a_{11})b_{1k} + (a_{i2} + r_i a_{12})b_{2k} + \cdots + (a_{in} + r_i a_{1n})b_{nk} = c_{ik} + r_i c_{1k}.$$

We can now establish the situation of our special case by choosing r_i in such a way that $a_{i1} + r_i a_{11} = 0$, $(i \neq 1)$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1116. *Proposed by C. W. Trigg, Los Angeles City College*

We define a pandiagonal heterosquare as a square array of the first n^2 positive integers, so arranged that no two of the rows, columns, and diagonals (broken as well as straight) have the same sum. Is there any n for which these $4n$ sums are consecutive numbers?

E 1117. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct a right triangle in which the legs and the altitude on the hypotenuse can be taken as the sides of another right triangle.

E 1118. *Proposed by Joel Brenner, State College of Washington*

Let us say a permutation is of type T if it is a product of mutually permutable transpositions. Is every permutation p a product of two permutations of type T?

E 1119. *Proposed by L. C. Graue, Sacramento State College*

Consider two families of circles, one tangent at the origin to the x-axis and the other tangent at the point (1, 1) to a line of slope m. Find the locus of the points of tangency of the two families.

E 1120. *Proposed by W. B. Carver, Cornell University*

(a) A student working a numerical problem gets the result

$$59\sqrt{90 - 14\sqrt{7}} + 4\sqrt{4555 + 1721\sqrt{7}},$$

but he finds the answer in the book, which happens to be correct, is

$$145\sqrt{26+2\sqrt{7}}.$$

Show that the student's result is correct.

(b) Find all sets of integers k, R, S, T, A, B, C, D, E, F which satisfy the equation

$$R\sqrt{A+B\sqrt{k}}+S\sqrt{C+D\sqrt{k}}+T\sqrt{E+F\sqrt{k}}=0,$$

with $k>0$ and having no square factors, $A+B\sqrt{k}>0$, $C+D\sqrt{k}>0$, $E+F\sqrt{k}>0$, and the radicals meaning in all cases the positive square root.

SOLUTIONS

Time Soliloquy

E 1086 [1953, 626]. *Proposed by T. A. Bickerstaff, University of Mississippi*

"That was a good lunch; now for a good cigar and then I must catch the one o'clock train. Let's see—my watch says exactly nine o'clock but that can't be right. It's still running and well wound. Now I remember I wound it and set it just this morning by the radio. Maybe I carelessly set the hands in reverse position. If so, exactly what time is it?"

Solution by C. F. Pinzka, Educational Testing Service, Princeton, N. J. Let x and y $(0\leq x<y<12)$ represent the positions of the hands when they were set in reverse position. If both the correct and the reverse positions are to be meaningful, we must have $12x-y\equiv 12y-x\equiv 0 \pmod{12}$, or $13(y-x)\equiv 0 \pmod{12}$. The error due to reversing the hands is $y-x=(12/13)k$, k being an integer. Only $k=4$ leads to a time that is reasonable, this time being $12{:}41\frac{7}{13}$.

Also solved by Leon Bankoff, A. P. Boblétt, Julian Braun, W. B. Carver, Monte Dernham, S. H. Eisman, L. R. Ford, A. H. Freitag and Herta Freitag (jointly), Vern Hoggatt, R. T. Hood, R. W. Huff, A. R. Hyde, R. Klopfenstein, Sam Kravitz, L. V. Mead, E. F. Myers, C. S. Oglivy, Azriel Rosenfeld, C. Swanson, Daniel Weiner, R. H. Wilson, Jr., and the proposer.

A Special Case of Stern's Formula

E 1087 [1953, 626]. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Let a be an arbitrary positive integer and let

$$S_m=\sum_{j=1}^{a} j^m.$$

We have

$$a(a+1)^2=S_1+3S_2, \qquad a^2(a+1)^3=S_2+2S_3+5S_4,$$
$$a^3(a+1)^4=S_3+5S_4+3S_5+7S_6, \cdots .$$

Determine coefficients c_k such that

$$a^n(a+1)^{n+1} = \sum_{k=n}^{2n} c_k S_k.$$

Solution by F. R. Olson, Duke University. We write

$$x^n(x+1)^{n+1} = \sum_{s=0}^{n+1} \binom{n+1}{s} x^{2n+1-s} \tag{1}$$

and

$$(x-1)^n x^{n+1} = \sum_{s=0}^{n} \binom{n}{s} (-1)^s x^{2n+1-s}. \tag{2}$$

Subtraction of (2) from (1) and summation over x from $x=1$ to $x=a$ gives

$$a^n(a+1)^{n+1} = \sum_{s=0}^{n} \left[\binom{n+1}{s} + (-1)^{n-s} \binom{n}{s-1} \right] S_{n+s}, \tag{3}$$

where we define $\binom{s}{-1} = 0$. This method may be applied to handle the more general case $a^n(a+1)^m$.

These results are also obtained in a different manner in Nielson, *Traité Élémentaire des Nombres de Bernoulli* (1933), p. 305.

Also solved by W. E. Briggs, Leonard Carlitz, P. L. Chessin, F. J. Duarte, S. H. Eisman, N. J. Fine, A. R. Hyde, Bernard Jacobson, C. W. Karns, M. S. Klamkin, Kovina Milosevich, M. J. Pascual, C. F. Pinzka, R. C. Read, D. C. Russell, E. P. Starke, Chih-yi Wang, and the proposer. Late solutions by Julian Braun and C. D. Olds.

Probability of Winning a Game

E 1088 [1953, 627]. *Proposed by M. J. Pascual, Siena College*

A and B agree to play n games and he who goes first has a chance of $a/(a+b)$ of winning that game. If the winner of any game goes first in the following game, then find (1) the probability of A winning the nth game if he goes first in the opening game, (2) the expected value of A's winnings in n games if each game is worth d dollars.

Solution by D. S. Greenstein, University of Pennsylvania. Let $P=a/(a+b)$, and let P_n denote the probability that A win the nth game. Then

$$P_{n+1} = P_n P + (1-P_n)(1-P) = (2P-1)P_n + (1-P), \qquad P_1 = 1.$$

These conditions are uniquely satisfied by $P_n = [1+(2P-1)^n]/2$. A's expected winning in the nth game is

$$P_n d - (1-P_n)d = (2P_n - 1)d = d(2P-1)^n.$$

His expected winning for the first n games, therefore, is given by

$$d\sum_{k=1}^{n}(2P-1)^k = d(2P-1)[1-(2P-1)^n]/2(1-P).$$

Also solved by P. H. Arnold, J. W. Baldwin, Julian Braun, S. H. Eisman, N. J. Fine, L. A. Fulk, R. E. Greenwood, J. M. Howell, A. R. Hyde, M. S. Klamkin, A. E. Livingston, C. F. Pinzka, Azriel Rosenfeld, William Small, and the proposer.

Greenwood and Klamkin pointed out that this problem appears in a more general context in Uspensky, *Introduction to Mathematical Probability*, p. 75.

Generalization of a Real Function Theorem

E 1089 [1953, 627]. *Proposed by W. R. Utz, University of Missouri*

Show that if $f(x)$ is a real function bounded below on $I=[a, b]$ and $g(x)$ is real and either (i) monotone increasing on I or (ii) continuous with $g(b)>g(x)$ for $a\leqq x\leqq b$, then given $\epsilon>0$ there is a constant $\lambda(\epsilon)>0$ such that $f(x)-\lambda g(x)$ cannot attain $\inf_{x\in I}[f(x)-\lambda g(x)]$ on $[a, b-\epsilon]$. (The special case $g(x)=x$ is used by E. Baiada, *Ann. Scuola Norm. Super. Pisa*, vol. 15 (1950), p. 111.)

Solution by L. E. Ward, Jr., University of Nevada. Let $M(\epsilon)$ denote the g.l.b. of $f(x)$ for $a\leqq x\leqq b-\epsilon$, and let $g(x_0)$ be the maximum of $g(x)$ for $a\leqq x\leqq b-\epsilon$. We are assured of the existence of x_0 if g is continuous, and, if g is monotone increasing, then $x_0=b-\epsilon$. Choose $\lambda(\epsilon)$ such that

$$\lambda(\epsilon) > \max\{[M(\epsilon)-f(b)]/[g(x_0)-g(b)], 0\}.$$

Since $g(x_0)-g(b)<0$, and since $f(x)\geqq M(\epsilon)$ and $g(x)\leqq g(x_0)$ for $x\leqq b-\epsilon$, we have

$$f(x)-\lambda g(x) > f(b)-\lambda g(b), \qquad x\leqq b-\epsilon.$$

Therefore, for $x\leqq b-\epsilon$, it follows that $f(x)-\lambda g(x)$ cannot attain $\inf_{x\in I}[f(x)-\lambda g(x)]$.

Also solved by R. C. Read and the proposer.

An Interesting Construction Problem

E 1090 [1953, 627]. *Proposed by B. M. Stewart, Michigan State College*

From one vertex of a triangle lines are to be drawn dividing the triangle into a set S of n triangles having equal inscribed circles.

(1) Show that in general the set S may be constructed by ruler and compass if and only if $n=2^s$.

(2) Show that the $n-k+1$ triangles formed by taking sets of k adjacent triangles of the set S have equal inscribed circles ($k=2, 3, \cdots, n-1$).

(3) Find a neat construction when $n=2$. (*Note.* From (2) it follows that repeated application of (3) will solve the problem when $n=2^s$.)

Solution by the Proposer. Given triangle ABC with the ordered points $C_0=A, C_1, \cdots, C_{n-1}, C_n=B$ on the side AB, let r_{ik} be the radius of the in-

scribed circle of triangle CC_iC_{i+k}, $i=0, 1, \cdots, n-k$. Let h be the length of the altitude drawn from C and let $\alpha=\tan A/2$, $\beta=\tan B/2$. We can show that the conditions $r_{01}=r_{11}=\cdots=r_{n-1,1}$ imply

$$r_{ik} = [1 - (\alpha\beta)^{k/n}]h/2. \tag{1}$$

On the one hand, when $k=1$, formula (1) shows by the Galois conditions for constructibility of the new quantity $(\alpha\beta)^{1/n}$ from the known quantities α and β, that (a) holds.

On the other hand, for any k, formula (1) is independent of i, which establishes (b).

To prove (1) we note that for any triangle such as ABC we may express the area in two ways involving h, r, α, β, and $\gamma=\tan C/2$, as follows:

$$(r/\alpha + r/\beta)h/2 = (r/\alpha + r/\beta + r/\gamma)r.$$

Since $\gamma=(1-\alpha\beta)/(\alpha+\beta)$, we obtain

$$r = (1 - \alpha\beta)h/2, \qquad (h - 2r)/h = \alpha\beta. \tag{2}$$

If we set $\alpha_i=\tan C_{i+1}C_iC/2$ and $\beta_i=\tan CC_iC_{i-1}/2$, then

$$\alpha_i\beta_i = 1, \tag{3}$$

because the external and internal angle bisectors are perpendicular. Applying (2) to the triangle CC_iC_{i+1} we have $(h-2r_{i1})/h=\alpha_i\beta_{i+1}$. Thus the conditions $r_{01}=r_{11}=\cdots=r_{n-1,1}$ hold if and only if

$$\alpha\beta_1 = \alpha_1\beta_2 = \alpha_2\beta_3 = \cdots = \alpha_{n-1}\beta.$$

Using (3) we readily check that this system of equations is solved by

$$\alpha_i = \alpha^{n-i}/\beta^i, \qquad i = 0, 1, \cdots, n-1.$$

For

$$(\alpha_i\beta_{i+k})^n = (\alpha^{n-i}/\beta^i)(\beta^{i+k}/\alpha^{n-i-k}) = (\alpha\beta)^k.$$

When $k=1$, this shows that the system of equations is solved; and for any k, this shows starting from (2) that

$$r_{ik} = (1 - \alpha_i\beta_{i+k})h/2$$

is equivalent to (1).

For part (c) we may let I be the incenter for triangle ABC and draw BQ parallel to IC to meet AC at Q. Then the circle through A, B, Q is the locus of points subtending an angle $C/2$ from AB. Extend IC to meet this circle at C'. From C draw CX parallel to $C'A$ to meet AI at X; from C draw CY parallel to $C'B$ to meet BI at Y. Since triangle XYC is homothetic to triangle ABC', it follows that X, Y are the centers of the desired equal inscribed circles, and their common tangent drawn from C will locate the desired point C_1 on the side AB.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4588. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Let $n>1$. Evaluate

$$\int_0^\infty (\sqrt[n]{x^n+1}-x)dx.$$

4589. *Proposed by Casper Goffman, Wayne University*

It is well known that the interval (0, 1) is the union of a set of the first category and a set of measure zero. Generalize this result to arbitrary separable metric spaces.

4590. *Proposed by Paul Erdös, University of Notre Dame*

Fermat's conjecture that all numbers of the form

$$F_n = 2^{2^n}+1$$

are prime was proved wrong by Euler. Show, however, that $\sum 1/d \to 0$ as $n\to\infty$, where d ranges over all the divisors of F_n except 1.

4591. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Let $f(x)$ be analytic. For $a=1$, the equation $f'(ax)+f(x)=0$ $(-\infty<x<\infty)$ has the property that its solution, $f(x)=e^{-x}$, approaches zero as $x\to\infty$. Is this true for any real $a>1$?

4592. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Find the sum

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}\log r}{r}.$$

SOLUTIONS

Postulates for a Group

4504 [1952, 554]. *Proposed by Olga Taussky, National Bureau of Standards, Washington, D. C.*

Prove that a set S which is closed under an associative composition law which satisfies the following three axioms is a group:

1. There exists an idempotent e such that $e^2=e$.
2. Every element has at least one left inverse with respect to e.
3. Every element has at most one right inverse with respect to e.

Note by D. W. Sasser, Branford, Connecticut. In the recent solution [1954, 55] the following statement is quoted as a "well known theorem": a semigroup with a right identity and left inverses with respect to it is a group. This is false as can be seen by the following example. Let S be any set with more than one element and define $ab=a$ for any two elements a, b in S. S is clearly a semigroup. Moreover, any element serves as a right identity and is in fact the left inverse of every element. But S is not a group since the identity is not unique—and for other reasons as well.

The statement is correct—and is easily proved—if one assumes that the right identity is unique. It is also true that: a semigroup with a right (or left) identity and with right (or left) inverses with respect to it is a group.

The proof as printed is correct when the alternate finish is followed.

Correction also noted by G. B. Preston.

Density of the Set of Sums of Two Squarefull Numbers

4525 [1953, 123]. *Proposed by Paul Erdös, University of Notre Dame*

Let $u_1<u_2<\cdots$ be the sequence of integers all of whose prime factors have exponents exceeding one. Prove that the density of integers of the form u_i+u_j is zero (*i.e.* the number of integers $\leqq x$ of the given form is $<\epsilon x$ for any ϵ if x is large enough).

Solution by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y. Borrowing from number theory the result that, for any fixed positive integers a, b, the set of numbers ax^2+by^2 has zero density, we are able to prove the general result:

If $\sum a_m^{-1/2}$, $\sum b_n^{-1/2}$ both converge then the sets of all integers of the form $a_mx^2+b_ny^2$ is of zero density.

From this the problem follows, as all squarefull numbers are of the form m^3x^2 and we need only apply the above result to the case where $a_m=m^3$, $b_n=n^3$.

Proof of result: We need only consider x, $y>0$. First note that the number of integers $\leqq N$ which are of the form Ax^2+By^2 is at most

$$\sqrt{\frac{N}{A}}\cdot\sqrt{\frac{N}{B}}=\frac{N}{\sqrt{AB}}.$$

Now split the integers $a_m x^2+b_n y^2$ into three sets

$$S_1: \quad m, n \leqq M; \qquad S_2: \quad m > M; \qquad S_3: \quad n > M.$$

Clearly S_1 has zero density (it is made up of a finite number of sets of the form ax^2+by^2). S_2 contains, among the first N integers, at most

$$\sum_{m,n;\, m>M} \frac{N}{\sqrt{a_m b_n}} = N \sum b_n^{-1/2} \sum_{m>M} a_m^{-1/2},$$

so that the upper density of S_2 is $\sum b_n^{-1/2} \sum a_m^{-1/2}$ with $m>M$. Likewise $\sum a_m^{-1/2} \sum b_n^{-1/2}$, with $n>M$, is the upper density of S_3. By letting $M \to \infty$ we see that S_2 and S_3 are of density zero, and so is our whole set.

Also solved by K. Prachar.

k-chromatic Graphs without Triangles

4526 [1953, 123, 336]. *Proposed by Peter Ungar, New York University*

Show that for any $k>1$ there exist k-chromatic graphs which contain no three mutually connected nodes. (If it requires k colors to color the nodes so that no two nodes of the same color are connected by an edge, the graph is k-chromatic.)

Solution by Blanche Descartes, Toronto, Ontario. A stronger result can be proved. It can be shown that for any $k>1$ there exists a k-chromatic graph which has no circuit of less than 6 edges.

The case $k=2$ is trivial. For the other cases we define a sequence of graphs $G_3, G_4, G_5, \cdots$. The graph G_3 is a circuit of just 7 edges. (Any larger odd number would do.) When G_i is defined, with m_i nodes say, we construct G_{i+1} as follows. We take

$$\binom{im_i - i + 1}{m_i}$$

disjoint copies of G_i. We adjoin im_i-i+1 extra nodes, which we call "central." We set up a 1-1 correspondence between the copies of G_i and the sets of m_i central nodes. We join each copy of G_i to the members of the corresponding set of central nodes by m_i new edges of which no two have a common end. The resulting graph is G_{i+1}.

This construction ensures that no graph G_i has a circuit of less than 6 edges.

Clearly G_3 is 3-chromatic. If $i>3$ and G_{i+1} has a coloring C in i or fewer colors then some m_i of the central nodes of G_{i+1} must have the same color in C. The corresponding copy of G_i must be colored in $i-1$ or fewer colors. We may thus show inductively that G_k cannot be colored in less than k colors. This does not prove that G_k is k-chromatic but if it is not we can obtain a k-chromatic graph from it by deleting some vertices and their incident edges. Thus for all $k>1$ there exists a k-chromatic graph having no circuit of less than 6 edges.

The writer discussed a problem equivalent to the special case $k=4$ in *Eureka* (March, 1948).

Also solved by J. B. Kelly and the Proposer.

Continuous Solution of a Functional Equation

4527 [1953, 123]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Is there a function $f(x)$ continuous in the closed interval (0, 1) and such that in this interval $f(x)+f(x^2)=x$?

I. *Solution by a reader whose name does not appear.** If the defining relationship is used iteratively the form of $f(x)$ is seen to be

$$f(x) = x - f(x^2) = x - x^2 + f(x^4) = \cdots$$
$$= x - x^2 + x^4 - \cdots + (-1)^k x^{2k} + (-1)^{k+1} f(x^{2k+1}).$$

The continuity of $f(x)$ at $x=0$ implies that $f(0)=0$ and therefore the last term above approaches zero as $k\to\infty$, so that

$$f(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

gives the unique solution on the open interval $0<x<1$ which is also continuous at $x=0$. If the desired $f(x)$ exists, it must be true also that $f(1)=\frac{1}{2}$.

Suppose there exists a point x_0, $0<x_0<1$, at which $f(x_0)=\frac{1}{2}+\Delta$, $\Delta>0$. Let $x_n=(x_{n-1})^{1/2}$. Then $x_n>x_{n-1}$ and $x_n\to 1$ as $n\to\infty$. From the defining relationship $f(x_n)=x_n-f(x_{n-1})$. In particular $f(x_1)=x_1-f(x_0)=x_1-(\frac{1}{2}+\Delta)<\frac{1}{2}-\Delta$, and $f(x_2)=x_2-f(x_1)=\frac{1}{2}+\Delta+(x_2-x_1)$. Since $x_2>x_1$, $f(x_2)>f(x_0)$. By induction, the points x_{2n} form a sequence approaching 1 as a limit such that

$$f(x_{2n+2}) > f(x_{2n}) > \cdots > f(x_2) > \tfrac{1}{2} + \Delta,$$

for all $n>0$. Similarly $f(x_{2n+1})<f(x_{2n-1})<\cdots<\frac{1}{2}-\Delta$. Thus $\lim f(x)$ as $x\to 1$ cannot exist.

Actually the value of $f(.995)$ by direct calculation exceeds 0.5008 which proves the impossibility of the proposed function.

II. *Solution by Norman Greenspan, Polytechnic Institute of Brooklyn, N. Y.* In G. H. Hardy, *Divergent Series*, p. 77, there is discussed the functional equation

$$f(x) + f(x^a) = x, \qquad a > 1,$$

in connection with the two distinct continuous solutions:

$$F(x) = \sum_{n=0}^{\infty} (-1)^n x^{a^n} \qquad 0 \leqq x < 1,$$

* Each month many contributions are received unsigned—eight for the single problem 4527. Wherever possible from attached notes or return addresses on envelopes, the editors add the names. We urge each author to make sure his name appears on every sheet submitted.

$$\Phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(1+a^n)} \left(\log \frac{1}{x}\right)^n \qquad 0 < x \leqq 1.$$

Hardy shows that $F(x)$ is not continuous at $x=1$ nor is $\Phi(x)$ continuous at $x=0$. Furthermore $F(x)$ and $\Phi(x)$ are the only functions of x that satisfy $f(x)+f(x^a) = x$ and are continuous for $0 \leqq x < 1$ and $0 < x \leqq 1$ respectively. Thus there exists no function of the type proposed.

Also solved by A. A. Bennett, P. M. Cohn, A. E. Currier, D. S. Greenstein and S. G. Kneale, Oliver Gross, P. G. Kirmser, Walter Knödel, H. W. Oliver, R. F. Reeves, E. M. Wright, and the Proposer.

Editorial Note. From the fact that $F(1)$ assumes the form $1-1+1-\cdots$, several readers assumed that $F(x) \to \frac{1}{2}$ as $x \to 1$. For further clarification of this point consult Levinson, *Gap and Density Theorems*, 1940, p. 186, for the theorem of Hardy-Littlewood; also Hardy, *Divergent Series*, p. 173, for the high-indices theorem.

Sequences with One Term Relatively Prime to All Others

4528 [1953, 192, 423]. *Proposed by Paul Erdös, University of Notre Dame*

Prove that the primes have the following property: If a sequence of consecutive integers $n, n+1, \cdots, n+k$ contains a prime p, then at least one of these integers is relatively prime to all others. Prove also that infinitely many integers which are not primes have the same property. (That sequences exist in which no member has this property has been shown by Sivasankaranarayana, Pillai and Szkeres.)

Partial Solution by Stanislaw Leja, Buffalo, N. Y. Let $n+i$ be the greatest prime contained in this sequence. Then $2n+2i > n+k$, for if not, then by Bertrand's postulate (for Tchebychef's proof see Landau, *Primzahlen*, v. I, pp. 89–92) there exists a prime q such that $n+i < q < 2n+2i \leqq n+k$, and this is contrary to the supposition. So $n+i$ is relatively prime to all others.

Solved also (partially) by W. E. Briggs, T. A. Brown, W. F. Cheney, Jr., D. C. B. Marsh, Leo Moser, C. R. Phelps, Azriel Rosenfeld and Seymour Haber.

Editorial Note. The problem as originally given [1953, 192] was restated because sequences lacking the property that at least one member is relatively prime to all others have already been given. S. S. Pillai (*Proc. of the Indian Acad. of Sci., Sec. A*, v. ii (1940) pp. 6–12) proved that every set with $k<16$ possesses the property, but sequences with $k \geqq 16$ exist which lack it. A. Brauer (*Bull. Amer. Math. Soc.* (1941), pp. 328–331) proved that sets which do not have the property exist for all $k \geqq 16$. The simplest such set, noted by several solvers, is 2184, 2185, $\cdots$, 2200.

It was the Proposer's hope that an elementary proof of the second part would be submitted. His own treatment depends upon the lemma: *There are more than*

$$\frac{c_1 x}{\log x \log \log x}$$

primes $p \leqq x$ such that all prime factors of $p-1$ are greater than $(\log x)^2$. The proof of the lemma is not elementary and is complicated. It uses Bruns' method and the fact that for every $d<(\log x)^k$ and $(a, d)=1$, the number of primes $p \equiv a \pmod{d}$, $p<x$ equals

$$(1 + o(1)) \frac{x}{\phi(d) \log x}.$$

It can be shown that for an infinite subset q_i of the primes described in the lemma, the numbers q_i-1 satisfy the requirements of the problem.

A Convergent Series whose Partial Sums Have Distinct Real Roots

4529 [1953, 192]. *Proposed by C. D. Olds, San Jose State College, California*

Can one find a convergent series $a_0+a_1x+a_2x^2+\cdots$, where the a's are real and positive and such that all the roots of each of the equations

$$a_0 + a_1x + a_2x^2 = 0$$
$$a_0 + a_1x + a_2x^2 + a_3x^3 = 0$$
$$\cdots\cdots\cdots\cdots$$

are real?

I. *Solution by S. H. Gould, Purdue University.* Suppose that the equation $a_0+a_1x+\cdots+a_nx^n=0$, with positive coefficients, has the n distinct real roots $r_1, r_2, \cdots, r_n$. Then

$$a_nx + a_{n-1}x^2 + \cdots + a_0x^{n+1} = 0$$

has the $(n+1)$ distinct real roots

$$r_1^{-1}, r_2^{-1}, \cdots, r_n^{-1}, 0. \tag{1}$$

Since the roots are continuous functions of the coefficients, the $(n+1)$ roots of

$$a_{n+1} + a_nx + \cdots + a_0x^{n+1} = 0$$

are, for sufficiently small $a_{n+1}>0$, arbitrarily close to (1), hence real and distinct. Their reciprocals are roots of

$$a_0 + a_1x + \cdots + a_{n+1}x^{n+1} = 0.$$

Thus the problem is solved by successively choosing a_{n+1} small enough to meet the above condition for $n=1, 2, \cdots$, and also small enough to ensure convergence of the series.

II. *Solution by I. N. Baker, Adelaide University, Australia.* Consider the particular example

$$f(x) = 1 + \frac{x}{4^1} + \frac{x^2}{(4^2)^2} + \cdots + \frac{x^n}{(4^n)^n} + \cdots$$

which converges for all x. For the nth degree polynomial

$$f_n(x) = 1 + \frac{x}{4} + \cdots + \frac{x^n}{4^{n^2}},$$

we can show that

(1) $\qquad f_n(-4^{4r}) > 0 \qquad$ when $n \geqq 2r$,

(2) $\qquad f_n(-4^{4r+2}) < 0 \qquad$ when $n \geqq 2r+1$.

In fact, we have

$$f_n(-4^{2r}) = \sum_{k=0}^{\infty} (-1)^k T_k, \qquad T_k = 4^{k(2r-k)}.$$

The largest T_k is T_r and the terms decrease in magnitude as we leave T_r on either side, being alternately positive and negative. The terms on either side are all numerically $\leqq T_r/4$. We conclude that the sums of terms to the left and right respectively of T_r are each numerically $\leqq T_r/4$. Therefore $f_n(-4^{2r})$ has the sign of the rth term, *i.e.* $+$ for r even, $-$ for r odd. This argument applies only when the term T_r appears on the right, *i.e.* when $r \leqq n$, or when the conditions (1) and (2) are satisfied.

Now consider the equation $f_n(x) = 0$.

(a) If n is even, $n = 2r$, we have

$$f_n(0) > 0,\ f_n(-4^4) > 0, \cdots, f_n(-4^{4r}) > 0,$$
$$f_n(-4^2) < 0,\ f_n(-4^6) < 8, \cdots, f_n(-4^{4r-2}) < 0,$$

whence there are n real roots r_i such that

$$0 > r_1 > -4^2 > r_2 > -4^4 > \cdots > -4^{4r-2} > r_n > -4^{4r}.$$

(b) If n is odd, $n = 2r+1$, we have $2r$ roots exactly as above, all $> -4^{4r}$. But we have also $f_n(-4^{4r}) > 0$, $f_n(-\infty) < 0$. Therefore there is one additional real root which is less than -4^{4r}. This makes up the full number of $n = 2r+1$ real (negative) roots.

The above proof also shows that $|f_n(-4^{2r})| > 2/4 \cdot 4^{r^2}$ and hence as $n \to \infty$, $f_n(-4^{2r})$ approaches a limit L of sign $(-1)^r$ and different from zero. Thus $f(x)$ is an entire function which vanishes at least once in every interval $(-4^{4(r+1)}, -4^{4r})$.

No significance is to be attached to the use of 4 as base. Any real number greater than 2 will serve.

Also solved by R. H. Breusch, Leonard Carlitz, Myles McConnon and L. F. Boron, and O. E. Stanaitis.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Math Is Fun. By Joseph Degrazia. New York, Emerson Books, Inc., 1954. xvii+159 pages. $2.75.

This book consists of trick problems, puzzles and, as the author says, "brain teasers." Its main purpose is to furnish mathematical recreation and to stimulate interest in mathematics. The book is divided into two parts. The first part contains clearly stated problems with little or no hint as to the solution. The second part is devoted to carefully written solutions to most of the problems.

The problems are classified into chapters which serve to facilitate the reader in finding the particular type in which he may be interested. Such chapters as Faded Documents, Cryptograms, Clock Puzzles, Speed Puzzles, Shopping Puzzles, Railroad Shunting Problems, Agriculture Problems, are suggestive of this classification. The book contains a great variety of problems ranging from the very simple to many that require logical thinking and very careful analysis for their solution. The amateur as well as the experienced mathematician will find plenty of food for thought.

There are one hundred seventy-nine problems classified into seventeen chapters. Some are quite old and well known to mathematicians, but there is an abundance of new ones. There are many illustrations which are unique and interesting as well as helpful in clarifying the meaning of some of the problems. This is especially true of the chapter on railroad problems and the one related to squares.

Since no knowledge of mathematics beyond arithmetic and elementary algebra is needed for the solution to most of the problems, many of them could be used to add interest to high school algebra and arithmetic. College classes in elementary mathematics will find much material for thought in this volume. The mathematician or layman looking for recreation in mathematics will find an abundance of material suitable for such pastime.

The author is to be congratulated on producing such a well classified volume. It should be an interesting addition to any high school or college library and to many private libraries.

S. W. MCINNIS
University of Florida

Stability Theory of Differential Equations. By Richard Bellman. New York, McGraw-Hill Book Company, Inc., 1953. xiii+166 pages, $5.50.

This book is concerned with real solutions of real differential equations and the behavior of these solutions as the independent variable increases without limit. The properties of the solutions of greatest interest are boundedness, asymptotic behavior, oscillation, and stability. Very little of the material is to be found in other textbooks since most of it is either original with the author or was taken by him from research papers in the literature.

There has long been a need for an up-to-date text in English to serve as a basis for a graduate course in the theory of ordinary differential equations. This book should help to meet this need as well as to stimulate others to carry on investigations of some of the unsolved problems in this very fascinating and increasingly important area of analysis. The book is not intended to be encyclopedic in scope but rather as introductory to the modern theory of stability and asymptotic behavior of linear and non-linear differential equations.

The book opens with a detailed treatment of the linear system using matrix methods. The results in matrix theory required are derived from the beginning assuming no previous acquaintance with matrix theory. After the linear system with constant and almost-constant coefficients has been treated in considerable detail, the results of the linear theory are used to derive the results of Poincaré and Liapounoff concerning the stability of nonlinear systems. A survey of the important results concerning the boundedness, stability, and asymptotic behavior of the second-order, linear differential equations is presented. The last two chapters are devoted to some important nonlinear differential equations whose solutions may be completely described as far as asymptotic behavior is concerned. Any discussion of periodic solutions of nonlinear equations is omitted since such a discussion would require the use of more advanced analytic and topological tools, and the author desires to keep the book at a somewhat more elementary level.

Considerable pain has been taken in the writing and organizing of this book. Each of the seven chapters is divided into sections with appropriate headings and is closed by a bibliography classified as to the section of the chapter concerned. The various definitions, theorems, lemmas and corollaries are carefully labeled and italicized. Proofs are presented in a pleasing and direct style, and a considerable number of well chosen exercises are distributed throughout the text. The book contains a minimum number of typographical errors. It should prove to be a most valuable text for the student of analysis as well as an excellent reference book for the student interested in applications.

J. J. L. HINRICHSEN
Iowa State College

Introduction to the Theory of Statistics. By Victor Goedicke. New York, Harper and Brothers, 1953. 12+286 pages. $4.50.

The first 130 pages of this 286-page textbook cover quite satisfactorily the objectives of statistics, the preparation of a frequency tabulation, the use of logarithms and the slide rule, the basic laws of probability, the fundamentals of the normal curve, and methods of computing such basic sample statistics as are usually used to describe central tendency, dispersion, skewness, and kurtosis. An introduction to the use of the method of moments is also included.

Chapter 8 devotes 15 pages to the fitting of a straight line to data by methods of least squares and master points, and two pages to curvilinear curve fitting. Throughout the book problems are interspersed at strategic points and the answers are immediately available in the back of the book. This should be helpful to anyone wishing to study by himself. Chapter 9 quite thoroughly covers simple linear correlation in 43 pages and paves the way for a discourse on multiple and partial correlation in Chapter 12. Chapters 10 and 11 delve into sampling and reliability, and the testing of statistical hypotheses. A final short chapter called "statistics and common sense" contains some excellent advice to beginning students (and others). However, this reviewer is not as willing as the author seems to be to believe that Professor Kinsey established the validity of his original sampling technique in his first book on sexual behavior through the method cited on page 259.

With scores of modern and up-to-date examples and illustrations of the practical use of simple correlation readily available, it was a surprise to find Chapter 9 on simple correlation starting off with a diagram of data showing the association between cricket chirps and temperature compiled in 1927, and also one of the association of legislative ability and brain weight from a study made in 1932.

It seems to be the standard practice in writing elementary books on statistics to wave the magic wand called "normal distribution" over a situation and then quickly get on with an array of techniques and computations in which the early assumption is soon lost. How often one reads a statement similar to the one on page 166: "If we assume that the errors of prediction, $y-y_p$, will be distributed normally, we can solve all such problems by means of the normal curve tables." Quite true! But this assumption made in the use of the standard error of estimate may easily lead a beginning student into the idea that all that is required is to make such an assumption, and then proceed forthwith to make an estimate from the regression line, backing up his estimate with a probability of correctness obtained from the table of areas under the normal curve. The feeling may be established that somehow through the magic of mathematics all will be well once that assumption of normality is made. It is not made clear in Chapter 9 that the assumption of normality takes a lot for granted, that the sample regression line is not the true regression line of the universe, and that the sample

standard error of estimate is not the true standard error of estimate of the universe. The problem of estimating from a curve takes on a different aspect when these points are understood.

In Chapters 10 and 11 are further illustrations of the use of the broad assumption of normality in the population without sufficient emphasis and restatement of the danger in the indiscriminate use of this assumption. Answers are not found in Chapters 9, 10, and 11 to such questions as: 1—how can one tell whether or not use of the assumption of normality is justified; 2—what alterations need to be made in the formulae or tables used when the normal assumption is not applicable; 3—what can one do to improve the situation if the underlying assumption cannot be used. More stress should be put on the importance of randomization of samples, homogeneity of data, and basic knowledge of the population sampled. Principal emphasis should obviously be put on situations in which we do not know either the population mean or the population standard deviation, but the student should be made aware of other situations in which some knowledge of these parameters is available.

A few minor points noted were: the use of the word "proved" seems a bit strong when used on pages 216–218 in connection with the testing of statistical hypotheses; perhaps a better word could be found. In more than a score of places throughout the book the word "data" is used in the singular. On p. 111, 10th line, table 6–91 should be 6–9–1. On p. 219, in line beginning "If we replace . . . ," radical signs were omitted over the two denominators. On p. 271, the 4th formula is 11–5–1, not 10–5–1. Finally, out of 36 textbooks from the reviewer's shelves this was the only one using a bar above all expressions of averages, one of four using small x-bar for the arithmetic mean. Lack of standardization in symbolism is one of the curses of communication in the language of statistics.

G. I. BUTTERBAUGH
University of Washington

Mathematics and Statistics for Economists. By Gerhard Tintner. New York, Rinehart and Company, Inc. 1953. xiv+363 pp. $6.50.

This text, by an authority in the field of econometrics, is addressed to the student of economics who, with only high school training in algebra and trigonometry, wishes "to acquire some of the mathematical equipment necessary nowadays, for a serious study of economics."

The book is divided into three main sections: (1) Some Applications of Elementary Mathematics to Economics, (2) Calculus, (3) Probability and Statistics. Part 1 contains short chapters on functions and graphs, linear and quadratic equations, logarithms, progressions, determinants, and linear difference equations with constant coefficients. Part 2 contains the usual introductory material on differentiation (including such topics as functions of several variables, partial differentiation, homogeneous functions) and concludes with a short chapter on integration. Part 3 discusses the definition and laws of prob-

ability, random variables, probability distributions, sampling theory, tests of statistical hypotheses, regression, correlation, and index numbers.

In order to cover this variety of topics the author has had to write concisely in the text proper. The reader is helped by the compensatory influence of many carefully worked out illustrative examples which, in each section, follow the text material and precede the list of exercises. It is clear that the author has taken great care in collecting exercises which serve not only to give practice in the routine manipulations involved in getting answers, but which at the same time amply demonstrate the wide usefulness of mathematics in economics. This is a theme that runs throughout the book. Such economic ideas as demand and supply functions, market equilibrium, marginal cost and revenue, elasticity of demand, consumer's surplus, as well as many others, are treated in both the text and exercises. That these exercises will attract the economist and make his study of mathematics more interesting is undeniable.

For the teacher who plans to use the book, this poses a serious problem. There will be a temptation to hurry through the text material in order to consider the exercises. We know that all too many elementary mathematics courses are nothing but problem solving sessions. A few definitions, an intuitive argument which "convinces" the student of the validity of a theorem or method, an illustrative example or two, and then an assignment which requires only that the student mimic these examples so as to obtain answers which check with those at the back of the book. Such a course, whether it be for engineers or economists, does not succeed in training students to apply mathematics intelligently in their own field. The basic subject matter must be given time. The logic and development of the subject must be considered and given its due place in the classroom and in the homework assignment. For this reason, the reviewer regrets to note that the care taken by the author in selecting exercises seems not to have been lavished on the text proper.

In this connection, the following items seem noteworthy: (i) The arithmetic in the example on p. 87 needs to be reworked. (ii) The requirement that $M \neq 0$ should be added in the statement of the proposition on p. 103. (iii) The infinite series for e makes a sudden appearance (p. 110) without any previous, or later explanation of what such an infinite series means. (iv) On p. 123 the author asserts that if to a positive increment Δx there corresponds a positive (negative) increment Δy, then y' must be positive (negative). The example considered immediately following is $y = x^2$, but although the slope is calculated for four different points, the consideration of the slope at $x = 0$, which shows the above statements to be false, is omitted. (v) The use of the word "means" in the sentence (p. 183), "The last expression in this formula is the definite integral, and it means $F(b) - F(a)$, $\cdots$" is especially misleading to the reader who has just seen, in the preceding line that this "expression" is *defined* as the limit of a certain sum. (vi) On p. 205 the author refers to the uniformly distributed random variable in the interval (2, 6) as the "random variable x which can assume all possible values between 2 and 6 with *equal* probability." The student's

difficulty in understanding what is involved here is compounded by the assertion which follows, *viz.*, "this circumstance yields the probability density, $p(x)=1/4$, $2 \leqq x \leqq 6$." The "proof" that is given merely shows that the definite integral of $p(x)$ from 2 to 6 is equal to one, *i.e.*, that $p(x)$ is *a* probability density, not that it is *the* probability density of x. One problem here is that the author has not defined or discussed probabilities of events for which the underlying sample space is not discrete. (vii) The independence of two events is mentioned, but the independence of random variables is not defined. Nevertheless, this concept is used (p. 213, p. 215). (viii) The statement concerning the meaning of the confidence limits obtained in example 1, p. 252, is incorrect.

An instructor experienced in both mathematics and economics can overcome such difficulties. This book should then prove to be a useful textbook, not only for the future econometrician but for all students of economics.

SAMUEL GOLDBERG
Oberlin College

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

THE INTERNATIONAL SCHOLARS FORUM

The International Scholars Forum has been organized to facilitate the publication of manuscripts likely to have a limited sale. The members of the Advisory Board of the Forum are as follows: Professor J. A. De Haas, Claremont Men's College; Director Philip Munz, Rancho Santa Ana Botanic Garden; Professor W. T. Jones, Pomona College; Professor Edward Weismiller, Pomona College; President Frederick Hard, Scripps College; Librarian David Davies, Honnold Library, Associated Colleges at Claremont.

The Board has entered into an agreement with Martinus Nijhoff of the Hague to receive manuscripts, appraise them, and make recommendations regarding publication. For further information write to Librarian of the Honnold Library, Claremont, California.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1954:

Duke University. June 9 to July 17: Professor Gergen, algebra from an advanced standpoint; Professor Dressel, advanced calculus I. July 20 to August 27: Professor Carlitz, advanced calculus II. July 19 to 23: laboratory conference for teachers of science and mathematics.

Kent State University. June 21 to July 30: Professor Olson, differential equations I; Professor Bush, introduction to modern algebra; Assistant Professor Brumfield, advanced methods of teaching mathematics in high school. August 2 to September 3: Professor Jenkins, probability; Professor Brooks, history of mathematics.

Michigan State College. June 23 to July 30: Professor Grove, advanced calculus I; Mr. Kraft, analysis of variance; Professor Nordhaus, differential equations, solid and spherical geometry; Professor Olkin, elements of statistics, correlation analysis; Professor Robinson, theory of numbers, foundations of geometry. June 23 to August 20: Professor Hannan, statistical methods in engineering; Professor Hill, advanced calculus III, complex variable I; Professor Katz, time series and index numbers, design of experiments; Professor Parkus, mathematics of engineering, vector analysis, boundary value problems.

Northwestern University. June 18 to August 10: Professor Holyoke, theory of equations, engineering mathematics III; Professor Curtis, differential equations; Professor Dwass, probability; Professor Reid, functions of a complex variable; Professor Boothby, the continuum.

University of California at Los Angeles. June 21 to July 30: Professor Coddington, functions of a complex variable; Drs. Motzkin, Teichroew and Wasow, seminar in numerical analysis; Professor Straus, partial differential equations of mathematical physics.

PERSONAL ITEMS

Los Angeles City College announces that the Mathematics Department of the College with the cooperation of the mathematics teachers and high school students of the Los Angeles City Schools is publishing a Mathematics Newsletter. The objective of the Newsletter is to stimulate interest in mathematics among high school students. Further information about the Newsletter may be obtained from Mr. Samuel Skolnik, Los Angeles City College.

The University of Cincinnati announces that Dr. Alexander Peyerimhoff of the University of Giessen has joined the University research group which is investigating the theory of Nörlund means and its applications under the sponsorship of the United States Air Force through the Office of Scientific Research of the Air Research and Development Command. Dr. Peyerimhoff will work with the group during the current semester. Dr. Wolfgang Jurkat of the University of Tübingen and Professors H. D. Lipsich and C. N. Moore of the University are continuing to work with the group. Professor Konrad Knopp served with the group during the first semester of 1953–54.

The University of Connecticut reports that a new Department of Statistics

has been established in the College of Arts and Sciences of the University. Professor Geoffrey Beall is Chairman of the Department.

Mr. James Bates, formerly a graduate research mathematician at Visibility Laboratory, Scripps Institute of Oceanography, is now a staff mathematician with Military Operations Research Division, Lockheed Aircraft Corporation, Burbank, California.

Mr. L. F. Boron has been appointed to an instructorship at the University of Colorado.

Assistant Professor Truman Botts of the University of Virginia has received a Ford Foundation Grant and is on leave for the academic year 1953–54.

Mr. B. A. Chiappinelli, previously a mathematician for the Rand Corporation, Santa Monica, California, is now with Hughes Aircraft Company, Culver City, California, as a business systems analyst.

Dr. E. P. Coleman, formerly visiting professor, has been appointed Professor in the Department of Engineering, University of California at Los Angeles.

Dr. M. S. Demos has been appointed to a professorship at Drexel Institute of Technology.

Mr. P. C. Fife, previously a graduate student at the University of California, is now a physicist with Sandia Corporation, Albuquerque, New Mexico.

Mr. E. I. Gale has accepted a position at Brockville Bible College.

Associate Professor E. L. Godfrey of Defiance College has been promoted to a professorship.

Mr. F. D. Grogan has been promoted from the position of Army Chemical Corps Resident Inspector at Shwayder Brothers, Inc., Denver, Colorado, to the position of Chief, Arsenal Inspection Section, Rocky Mountain Arsenal, Denver, Colorado.

Mr. F. P. Harding, formerly a student at Marquette University, is teaching at Clarenceville High School, Livonia, Michigan.

Mr. J. E. Householder, previously a graduate student at the University of Arizona, has been appointed to a part-time instructorship at the University of Colorado.

Dr. M. A. Hyman, who returned recently from Holland where he was a Fulbright Scholar, has accepted a position in the Atomic Power Division of the Westinghouse Electric Corporation, Washington, D. C.

Dr. E. B. Leach has been appointed to an instructorship at Case Institute of Technology.

Associate Professor Mary A. Lee of Sweet Briar College is on leave of absence and has accepted a position as Assistant Mathematician at the Rand Corporation.

Mr. Stanislaw Leja, previously with the Ford Motor Company, Buffalo, New York, has been appointed to an instructorship at Cornell University.

Dr. John McCarthy of Princeton University has been appointed to an acting assistant professorship at Stanford University.

Mr. G. W. Medlin of Wake Forest College has been promoted to an assistant professorship.

Professor C. T. Molloy of Polytechnic Institute of Brooklyn has accepted a position as Head of Physics Research, Vitro Corporation of America.

Dr. E. S. Northam of Michigan State College has a position as a mathematician with the Bendix Aviation Corporation, Detroit, Michigan.

Mr. D. G. O'Connor has been promoted to the position of Technical Engineer in Planning Engineering, International Business Machines Corporation, Endicott, New York.

Dr. R. H. Owens of Brown University has accepted a position as Physical Sciences Coordinator with the Office of Naval Research, Pasadena, California.

Assistant Professor R. P. Peterson of the University of Washington has been appointed to an assistant professorship at the University of California, Riverside.

Professor Emeritus George Pólya of Stanford University has been appointed to a visiting professorship at the Swiss Federal Institute of Technology.

Miss Filomena R. Reyes has been appointed to an instructorship at the University of the East, Manila, Philippines.

Mrs. Joy Russek, previously at New York University, has been appointed to an instructorship at the University of Buffalo.

Dr. D. E. Sanderson has been appointed to an instructorship at Iowa State College of Agriculture and Mechanic Arts.

Associate Professor M. E. Shanks of Purdue University has been promoted to a professorship.

Mr. M. M. Slotnick, previously chief mathematician with Humble Oil and Refining Company, Houston, Texas, is now Consultant Geophysicist with Standard Vacuum Oil Company, New York City.

Associate Professor C. D. Smith of the University of Alabama has been promoted to a professorship.

Assistant Professor Frances M. Suter of Roanoke College is now Registrar at Peace College.

Professor A. W. Tucker of Princeton University gave a course in Combinatorial Topology in the autumn of 1953 at Haverford College under the Philips Bequest.

Mr. B. K. Youse, previously a graduate assistant at the University of Georgia, has been promoted to an instructorship.

Professor Emeritus J. L. Coolidge of Harvard University died on March 5, 1954. He was a charter member and a former president of the Association; he also served as vice-president, as editor of this MONTHLY and as a member of the Board of Governors.

Professor Emeritus L. E. Dickson of the University of Chicago died on January 17, 1954. He was a charter member of the Association and an honorary life member since 1941.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-first annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Tulane University, New Orleans, Louisiana, on February 19 and 20, 1954. Professor M. E. Gillis, Chairman of the Section, presided at the afternoon session. Professor B. O. Van Hook presided at the evening session and Professor M. M. Ohmer presided at the Saturday morning session.

There were one hundred eight persons registered including the following sixty-three members of the Association:

R. E. Allan, T. A. Bickerstaff, W. H. Brothers, Jr., Elsie T. Church, W. H. Cleveland, G. J. Corley, Myrtis Davis, Margaret R. Davis, M. P. Dossey, W. L. Duren, Jr., D. O. Etter, P. L. Ford, Elizabeth Freas, L. M. Garrison, M. E. Gillis, F. L. Griffin, J. S. Griffin, Jr., A. C. Grimes, J. P. Gwin, Jessie M. Hoag, S. T. Hu, John Jones, Jr., L. H. Kanter, H. T. Karnes, C. G. Killen, Margaret M. LaSalle, Z. L. Loflin, Saunders MacLane, A. C. Maddox, J. W. McClimans, Betty McKnight, R. A. Miller, B. Ernest Mitchell, B. Evans Mitchell, D. E. Morrill, D. R. Morrison, S. B. Murray, M. M. Ohmer, Arthur Ollivier, R. L. O'Quinn, B. J. Pettis, T. J. Pignani, W. G. Preble, P. K. Rees, T. L. Reynolds, F. A. Rickey, L. A. Rife, A. A. Ritchie, Wm. M. Sanders, D. R. Scholz, H. F. Schroeder, L. L. Scott, S. W. Shelton, Jr., A. L. Shields, W. H. Spragens, Jr., R. A. Stokes, V. B. Temple, W. B. Temple, W. E. Timon, Jr., B. B. Townsend, Eugenia Trapp, B. O. Van Hook, A. D. Wallace.

The following officers were elected for the coming year: Chairman, Professor J. W. McClimans, Southeastern Louisiana College; Louisiana Vice-Chairman, Professor P. K. Rees, Louisiana State University; Mississippi Vice-Chairman, Professor R. A. Miller, University of Mississippi; Secretary-Treasurer, Professor Z. L. Loflin, Southwestern Louisiana Institute.

The invited speaker for the meeting was Professor F. L. Griffin, Newcomb College. His address on Friday evening was entitled "Some Mathematical Sectors of Biology." At the Saturday morning session he spoke on the topic "Is Mathematics a Science?"

The following papers were presented:

1. *A Diophantine matrix equation*, by Professor John Jones, Jr., Mississippi Southern College.

M. H. Ingraham and H. C. Trimble considered the matrix equation $XA=BX+C$. W. V. Parker considered the matrix equation $AX=XB$. W. E. Roth found a solution to the matrix equation $AX^2+BX+XC=D$, and also found a necessary and sufficient condition for a solution to matrix equations of the types $AX-YB=C$, and $AX-XB=C$. The purpose of this paper is to find a necessary and sufficient condition for a Diophantine matrix equation of the type $A(\lambda)X(\lambda)-Y(\lambda)B(\lambda)=C(\lambda)$ to have a solution, $X(\lambda)$, $Y(\lambda)$, where the elements of the n by n matrices $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $X(\lambda)$, and $Y(\lambda)$ are analytic functions of a complex variable λ in a region R.

2. *Unitary multiples of a matrix*, by Professor B. Evans Mitchell, Louisiana State University.

A canonical form for one-sided unitary multiples of a matrix (the analogue of the Hermite

form) is developed. Two-sided unitary multiples of a commutative set $A_1, \cdots, A_m$ of matrices are considered briefly in connection with Frobenius' theorem on the characteristic roots of $f(A_1, \cdots, A_m)$.

3. *A note on the zeros of the ultraspherical polynomials*, by Professor L. H. Kanter, Mississippi State College.

Let $P_n(x, \lambda)$ denote the ultraspherical polynomial of degree n, and let $x_\nu(\lambda)$ denote the νth zero of $P_n(x, \lambda)$. By making use of a well-known result of Stieltjes on the sign of the derivative of $x_\nu(\lambda)$, certain separation theorems for the zeros of $P_n(x, \lambda)$ and its parametric derivative (with respect to λ) are obtained.

4. *Characterization of knots*, by Professor Margaret M. LaSalle, Southwestern Louisiana Institute.

If the equations of the crossings of a knot are given, the knot diagram can be constructed and if the knot diagram is given the equations of the crossings can be obtained. The ϵ-equivalence of matrices is defined and is used as a criterion for knot equivalence (J. W. Alexander, *Topological Invariants of Knots and Links*, Trans. A. M. S., v. 30, 1928, pp. 275–306). This work was done in connection with the Topology Seminar under Professor Young at the University of Michigan, Summer Session, 1953.

5. *A hybrid system of coordinates*, by Professor B. Ernest Mitchell, University of Mississippi.

A hybrid system of coordinates is obtained when we select the abscissa of the Cartesian system and the radius vector of the polar system. The frame of reference is also hybrid. In this system asymmetry lends itself to simplicity rather than symmetry in each of the original systems. In a certain position of asymmetry the conic has an equation of the first degree, while a straight line in general position is of the second degree. The system is tied in with a certain conservative field of force.

6. *Applications of the scalar product in geometry*, by Mr. D. E. Morrill, University of Mississippi.

1. The vector equation $\lambda \cdot \rho = c$ exhibits a pole-polar relationship relative to a circle, center, the point of reference, radius $\sqrt{c}$; λ is a line vector (not a line bound vector), ρ is a point vector.

2. $(\rho - \alpha) \cdot (\rho - \beta) = 0$ is the equation of a circle constructed on the line adjoining the termini of α and β as a diameter.

3. $\rho \cdot \sigma = c$ is the equation of a congruence of circles, to wit, the circles orthogonal to the circle, center the point of reference, radius $\sqrt{c}$. The congruence is hyperbolic when $c > 0$, parabolic when $c = 0$, and elliptic when $c < 0$.

7. *Skidding cycloids*, by Professor V. B. Temple, Louisiana College.

The general equations of the hypo- and epicycloids can be written respectively in parametric form as:

$$x = \frac{a}{n}\left[(n \mp 1)\cos\theta \pm \cos\left(\frac{n}{\lambda} \mp 1\right)\theta\right],$$

and

$$y = \frac{a}{n}\left[(n \mp 1)\sin\theta - \sin\left(\frac{n}{\lambda} \mp 1\right)\theta\right],$$

where a is the radius of the directrix circle, $n > 1$, λ a constant $\neq 0$, and θ the parametric angle.

If: $\lambda=1$, we have the pure cycloids, positive and non-skidding;
$\lambda=-1$, the negative of the pure cycloids, and skidding;
$\lambda>1$ and $\lambda<-1$, + and − skidding respectively, and lagging;
$0<\lambda<1$ and $-1<\lambda<0$, + and − skidding, and advancing.

In particular if:

$2\lambda=\pm n$, we have lagging hypo- and epi-ellipses;
$\lambda=\pm n$, we have lagging hypo- and epi-circles.

8. *A non-clanoidal space*, by Professor A. D. Wallace, Tulane University.

Let X denote the "curve" $y=\sin x^{-1}$, $0<x\leqq(2\pi)^{-1}$, together with the segment from $(0, -1)$ to $(0, 1)$ on the axis of ordinates. It is shown that X does not admit a continuous associative multiplication with unit.

9. *Two analytic geometries for freshmen*, by Professor W. L. Duren, Jr., Tulane University.

The conventional analytic geometry is the Euclidean geometry of the plane, that is, the study of the invariants under Euclidean motion. There is another geometry of the plane, which is usually called "graphs," which may be described technically as the cartesian product of the affine lines and the affine line. This is the study of the invariants of the cartesian plane under the group of affine transformations:

$$x' = ax + b$$
$$y' = cy + d, \qquad a \neq 0, c \neq 0.$$

The second geometry is actually more prevalent in elementary mathematics than Euclidean geometry. In this paper a few results from these geometries are compared.

10. *Some mathematical sectors of biology*, by Professor F. L. Griffin, Tulane University.

While the life sciences are predominantly non-mathematical, numerous fragments of their literature embody uses of mathematics at the level of calculus or above. Roughly classified these fall into the following ten fields: metabolic or biochemical problems, growth and senescence, genetics, population problems, light biology, biophysical problems, anatomy and biometry, forest and agricultural experimentation, neurology, psychometrics. Professor Griffin listed many instances of these sorts with typical formulas employed in the literature, and commented on some of the more interesting and significant investigations.

11. *Is mathematics a science?*, by Professor F. L. Griffin, Tulane University.

A number of writers, notably the late Professor C. J. Keyser, have held that mathematics is not a science but is identical with formal logic, having no subject matter except propositional forms and their relations. In Professor Griffin's opinion the subject matter of mathematics includes also concrete systems like pure number and conceptual space, used as objectifications or models of abstract systems. In creating a new body of subject matter defined essentially by its postulates, a mathematician acts in the same manner as an artist; in exploring the further properties of an already created body of subject matter, a mathematician proceeds in the same manner as a natural scientist. Abstract deductions have great value for the study of concrete systems.

Z. L. LOFLIN, *Secretary*

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The MONTHLY is devoting this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Wash.) before the first day of the month preceding the issue in which the notice is to appear. Announcements should indicate the academic rank or similar description of the opening, but should not mention a specific salary. Blind ads are permissible which direct replies to a specifix box number in care of the Mathematical Association of America, Buffalo 14, N. Y. In order to conserve space and achieve uniformity, the privilege is reserved to rearrange advertisements. Advertisers will be billed by the Association at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.

ILLINOIS, Knox College, Galesburg, May 14–15, 1954.

INDIANA, Rose Polytechnic Institute, Terre Haute, May 1, 1954.

IOWA, Iowa State College, Ames, April 30–May 1, 1954.

KANSAS

KENTUCKY, University of Kentucky, Lexington, May 8, 1954.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Maryland, College Park, May 1, 1954.

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, Hamline University, St. Paul, May 8, 1954.

MISSOURI, University of Missouri, Columbia, May 7, 1954.

NEBRASKA

NORTHERN CALIFORNIA

OHIO

OKLAHOMA, Oklahoma City University, October 29, 1954.

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.

PHILADELPHIA, Princeton University, Princeton, New Jersey, November 27, 1954.

ROCKY MOUNTAIN, Colorado Agricultural and Mechanical College, Fort Collins, April 30–May 1, 1954.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN

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UPPER NEW YORK STATE, College for Teachers at Albany, May 1, 1954.

WISCONSIN, State Teachers College, Eau Claire, May 8, 1954.

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